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Essays in Microeconomic Theory

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This thesis is submitted for the degree of Doctor of Philosophy.

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To William B. S. Trimble (1921–2008)

Essays in Microeconomic Theory
Alan Michael Walsh

Abstract

We present a collection of three essays exploring topics in microeconomic theory: conflict, alliances, and the origins of society; supply chain networks and industrial organisation; and game theory on economic networks.

Chapter 1

Anthropological evidence has shown that humans in the earliest agricultural societies worked harder and had lesser health outcomes than humans in hunter-gatherer societies. We develop a model where hunting and gathering is more productive than agriculture, yet individually rational actors coordinate on a less productive agricultural equilibrium. In an agricultural society, a group of warriors with dominant fighting skills threaten hunters into subjugation and tax farmers a portion of their produce. We develop three submodels: a simple model where all agents are worse off than in a hunter-gatherer society, a model with inequality where warriors improve their payoff relative to hunting and gathering at the expense of all other agents, and a dynamic model describing the transition from a hunter-gatherer society to an agricultural society.

Chapter 2

Barriers to trade can create price discrepancies between markets. We apply this concept to an intermediation network, where the price at each node varies inversely with the quantity of resource supplied. We model a directed multipartite graph of intermediaries between a source and a market, where intermediaries in each partition simultaneously compete in the manner of Cournot competition, selecting the quantity of resource sold along each of their out-links. The linking structure represents each intermediary's opportunity to sell the resource. We derive an analytical solution determining the quantity decisions of each intermediary in the network, which we believe is the first such solution for a Cournot-driven supply chain. We discuss the efficiency of networks, and develop a measure that evaluates networks according to the consumer surplus received at the market.

Chapter 3

A set of agents is connected by two distinct networks, with each network describing access to a different local public good. Agents choose in which networks to invest, and neighbouring agents' investments in the same good are strategic substitutes, as are an agent's two investment choices. There are always equilibria where any investing agent bears all local investment costs and others free-ride. When investment in one good reduces marginal benefit from investment in the other, agents free-riding in one good may invest more profitably in the other, and equilibrium payoffs are more evenly distributed. This need not reduce aggregate payoff.

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Chapter 1

Violence and the Neolithic Revolution

1.1 Introduction

For upwards of 300,000 years, humans relied primarily on hunting and gathering for survival. Relatively recently, about 10,000 years ago, groups of humans began cultivating grains for food for the first time, a process termed the 'Neolithic Revolution' by Childe (1936). This was the first step in a series of technological and sociological developments that led to rapid population growth, urbanisation, and the formation of the advanced societies that we live in today.

We seek to evaluate the very first steps of agricultural development from an economic perspective. Why might Neolithic humans, who were thriving as a species, have undertaken such a significant change in their social structure? We are drawn particularly to the paradox, first discussed in Lee and DeVore (1968) and later developed by Sahlins (1972) and Cohen (1989), that the transition to agriculture made life more challenging for Neolithic humans. This has been demonstrated through the archeological study of human skeletons (Steckel 2004), and is a general consensus amongst anthropologists. Modelling Neolithic humans as rational decision makers, we develop a model where a failure to coordinate may result in self-interested actors choosing agriculture over hunting and gathering, despite the agricultural outcome ensuring lower utility for all participants.

The model is driven by violence, or the threat of violence, between agents. We assume that hunter-gatherers are do-it-all types, who both produce their own food for consumption and are responsible for their own protection during violent interactions. An agricultural society, however, features two types of specialists: farmers who produce food, and warriors who practise fighting skills and are responsible for the group's protection. We assume that warriors, as specialists who devote their entire allotment of time to the martial arts, have fighting skills that surpass those of hunter-gatherers.

Within an agricultural society, warriors require farmers to produce any food that they consume, and they are thus incentivised to oversee as many farmers as possible. To reach this end, warriors will seek out and attack hunter-gatherers who are not under their purview, driving down their chance of survival. Faced with this threat, hunter-gatherers will face the choice of continuing to hunt with a greater risk of death, or joining the agricultural society as farmers, where they would consume less but they will live a full life. As the fighting superiority of warriors over hunter-gatherers grows, the danger faced by hunter-gatherers grows, and they are more inclined to choose farming.

We show that our model may produce two types of equilibria. There will always be a hunter-gatherer equilibrium, where all agents choose to be hunter-gatherers. As well, there may be an agricultural equilibrium where all agents are either warriors or farmers. The existence of an agricultural equilibrium depends on the fighting superiority of warriors over hunter-gatherers, requiring that warriors have a sufficiently large edge in combat. In keeping with anthropological observations on the welfare of Neolithic humans, we assume that hunter-gatherers will always produce food more efficiently than farmers. Under this assumption, payoffs from the hunter-gatherer equilibrium will exceed payoffs from the agricultural equilibrium for all players.

To explore how society may have transitioned to agricultural equilibria, we develop an extension where warriors have more control over the equilibrium parameters. We make the technology of violence exclusive, meaning that neither farmers nor hunter-gatherers may transition to the warrior class. Then, the warriors face an optimisation problem, having control over the size of their group and the manner in which food is shared between farmers and themselves. With this additional control, the warriors form a smaller, more exclusive group, and exploit the farmers for a disproportionate share of resources. While aggregate utility in the resulting equilibrium is always lesser than in a hunter-gatherer equilibrium, under some parameterisations the warriors will achieve greater utility than they would as hunter-gatherers in the hunter-gatherer equilibrium. Thus, we may postulate a sequence of events that would have allowed the Neolithic Revolution to occur. Once a small group of warriors developed a technological ability to exert violent domination over others, they may have carried out a regime of violence to ensure such a transition took place.

There is little economic research exploring specifically the economic conditions in place prior to the Neolithic Revolution that may have led to the transition. Smith (1975) suggests that excessive hunting, particularly of large, easy-to-kill mammals, may have driven down the the marginal benefit from hunting to the point where it lay below that of farming. Similarly, North and Thomas (1977) focus on early property rights, arguing that the stock of animals available to hunt was common property for all with access and rife for exploitation and overuse. Agriculture, by contrast, featured private property rights, incentivising early farmers to extract food from their land at sustainable levels. Locay (1989) argues that the sedentary aspects of agriculture, as opposed to hunting which is more nomadic, may have increased parents' ability to rear healthy offspring in the face of increasing population pressures. Olsson and Hibbs (2005) hypothesise that changes in environmental conditions led to the transition to agriculture, and that local variations in environmental factors explain why different societies transitioned at different times. In the existing economic literature, economic factors—either endogenous in the cases of Smith, North and Thomas, and Locay, or exogenous in the case of Olsson and Hibbs—reduce the marginal benefit from hunting below that of farming. By contrast, in our model the activity of hunting and gathering always produces food in greater abundance than farming, but conflict between agents causes agents to settle into an agricultural outcome. Weisdorf (2005) provides a survey of the existing economic literature in this area.

A broader economic literature exists evaluating the consequences of the Neolithic Revolution on the institutions and social structures that later developed. Our model complements this work, providing a potential explanation for the first steps in this process. Galor and Moav (2002) introduce an economic growth model that attempts to capture all growth experienced since the Neolithic Revolution. Galor and Moav (2007) expand upon this theory, empirically linking variations in life expectancy in human populations to the time elapsed since the transition to agriculture. Similarly, Putterman (2008) links the timing of the Neolithic Revolution to incomes in the current day. Mayshar et al. (2011) argue that the transition to agriculture introduced the technology of taxation, which led to the development of more sophisticated government institutions.

The anthropological perspective on the origins of the Neolithic Revolution is too broad to fully survey here, but we provide a brief overview. There are two main schools in this literature. The first school holds that early humans responded to *opportunity*, where an event occurs that increases the yield from agriculture. The second school holds that the Neolithic transition was driven by *necessity*, where an event occurs that decreases the yield from hunting and gathering. Both theories converge on the requirement that agricultural yields must have exceeded the yields from hunting and gathering, but this premise has been contested by scholars such as Cohen (1989), Cohen and Armelagos (1984), Lee and DeVore (1968), Sahlins (1972) and Steckel (2004). Popular accounts of human development from the Neolithic Revolution through to the present day include Diamond (1997), Fernandez-Armesto (2001), Harari (2014) and Scott (2017).

Our model intersects with economic work on the study of conflict, and the tradeoff between productive and combative activities. Agents choose between hunting and gathering, which offers a high return but no protection, farming, with a lower return but the promise of protection, or fighting, which discourages hunter-gatherers, protects farmers, but has no productive return. This choice is similar to that faced by the participants in Hirshleifer's (1995) anarchic society, where participants must find a balance between productive or fighting effort. Grossman and Kim (1995) build a similar model, adding the distinction between appropriative and defensive activities. The relationship between conflict and equality in a society is developed by Esteban and Ray (1999). The extension of our model offers a binary exposition of this relationship: the hunter-gatherer equilibrium is perfectly egalitarian, while the warrior dominated agricultural society features a wealthy warrior elite and a poorer subclass of farmers. In these models conflict is introduced through a Tullock contest (Tullock 1980), a means of apportioning the spoils from any conflict. Garfinkel and Skaperdas (2007) summarises the economics of conflict literature and describes the use of Tullock contests. By contrast, in our model conflict is a winner-take-all contest that is always won by the more dominant party.

In work that is thematically similar to our contribution, Bó et al. (2019) explore how conflict helped encourage the development of civilisation in the Neolithic Era. However, their model begins with the assumption that environmental factors allow select societies to obtain a greater yield from agriculture than from hunting and gathering. These societies then invest in fighting technologies, in a similar manner to Grossman and Kim (1995), in order to prevent other societies from stealing their excess agricultural production. In contrast, our model explores how violence may allow agricultural societies to form when agricultural yields do not exceed the yield from hunting and gathering.

The chapter proceeds as follows: Section 1.2 introduces the basic model. Section 1.3 analyses equilibrium and compares aggregate welfare across different equilibria. Section 1.4 introduces an amendment to the model with restricted entry to the warrior class. In addition, we discuss how the amended game allows for a dynamic transition from a hunter-gatherer equilibrium to an agricultural equilibrium. Section 1.5 analyses comparative statics, focussing on the technology of violence available to the warrior types. Section 1.6 discusses how our model relates to previous economic literature on the Neolithic Revolution. Section 1.7 concludes. All proofs are presented in Appendix A.1.

1.2 Model

There is a unit continuum of agents, with each agent belonging to exactly one of three types, $\Theta = \{\theta_h, \theta_f, \theta_w\}$. We use A to denote the set of agents. The state of the economy is described by the vector $\boldsymbol{\mu} = (\mu_h, \mu_f, \mu_w)$, where μ_i is the mass of agents of type θ_i. As each agent is exactly one type, the set of states is equivalent to Δ^2, the standard 2-simplex in $\mathbb{R}^3$. We denote the set of agents of type θ_i, for $i \in \{h, f, w\}$, as A_i.

Each type represents an actor in a Neolithic economy. These are

Hunter-gatherers (θ_h): Hunter-gatherers subsist by hunting animals and gathering naturally occurring food. The return from this action is $v \in \mathbb{R}^+$. Hunter-gatherers may be threatened by warriors, and their probability of survival is $\big(1 - \phi(\mu_w)\big)$. If a hunter-gatherer does not survive, we assume they receive zero utility. The expected utility of each hunter-gatherer is

$$u_h(\boldsymbol{\mu}) = v\big(1 - \phi(\mu_w)\big)\,. \tag{1.1}$$

Farmers (θ_f): Farmers subsist by planting and cultivating crops. The return from this action is $\ell \in \mathbb{R}^+$. In addition, each farmer must pay the warriors a tax of t. The utility of each farmer is

$$u_f(\boldsymbol{\mu}) = \ell - t. \tag{1.2}$$

Warriors (θ_w): Warriors subsist on the taxes they receive from farmers. Warriors also use their fighting skill to threaten hunter-gatherers, which reduces the survival rate for hunter-gatherers and encourages them to convert to farming. The total tax revenue received from all farmers is distributed evenly between all warriors. The utility of each warrior is

$$u_w(\boldsymbol{\mu}) = t\frac{\mu_f}{\mu_w}. \tag{1.3}$$

The tax, $t \in [0, \ell]$, determines how the returns from farming are shared between type-θ_f and type-θ_w agents. As t increases, the size of transfer per type-θ_f agent increases. This will increase $u_w(\boldsymbol{\mu})$ while decreasing $u_f(\boldsymbol{\mu})$.

The *technology of violence*, $\phi : [0, 1] \to [0, 1]$, measures how effectively type-θ_w agents threaten the survival of type-θ_h agents. The function ϕ is continuous and increasing, with $\phi(0) = 0$ and $\phi(1) = 1$. $\phi(\mu_w)$ is the likelihood that a type-θ_h agent will not survive, given there is a mass of μ_w type-θ_w agents in the population.

All agents are expected-utility maximisers; that is, if $u_i(\boldsymbol{\mu}) > u_j(\boldsymbol{\mu})$, for $i, j \in \{h, f, w\}$, then agents will prefer choosing type θ_i over type θ_j. If $u_i(\boldsymbol{\mu}) = u_j(\boldsymbol{\mu})$, then agents will be indifferent between choosing types θ_i and θ_j.

Anthropological studies suggest that humans in the earliest agricultural societies experienced lesser health outcomes relative to the outcomes experienced by humans in hunter-gatherer societies. For this reason, the following assumption is made throughout.

Assumption 1.1. *$v > \ell$.*

Assumption 1.1 implies that, in the absence of a threat from type-θ_w agents, $u_h(\boldsymbol{\mu})$ is always greater than $u_f(\boldsymbol{\mu})$, and an individual agent will always prefer hunting and gathering over farming.

1.3 Equilibrium

In this section, we consider the situation where any agent $i \in A$ may freely choose his type $\theta_i \in \{\theta_h, \theta_f, \theta_w\}$. An equilibrium is defined by the vector $\boldsymbol{\mu}$ where no agent would prefer to be of a different type.

Definition 1.1. *The vector $\boldsymbol{\mu}$ describes an equilibrium if and only if the following condition is met:*

$$\mu_i > 0 \implies u_i(\boldsymbol{\mu}) \geq u_j(\boldsymbol{\mu}) \,\forall\, i, j \in \{h, f, w\}. \tag{1.4}$$

We define three potential types of equilibria: *hunter-gatherer equilibria*, where $\boldsymbol{\mu} = (1, 0, 0)$, *agricultural equilibria*, where $\boldsymbol{\mu} = (0, x, 1 - x)$, for $x \in [0, 1]$, and *mixed* *equilibria*, where $\mu_i > 0$ for all $i \in \{h, f, w\}$. We begin by claiming that there always exists a hunter-gatherer equilibrium.

Proposition 1.1. *For all v, ℓ, t and $\phi(\cdot)$, there exists a hunter-gatherer equilibrium.*

Proposition 1.1 is congruent with the observation that when humans first formed agricultural societies their health outcomes decreased. However, our interest lies in whether other equilibria may exist, which would have allowed the Neolithic Revolution to occur.

In the proof of Proposition 1.1, we show that if either $\mu_w = 0$ or $\mu_f = 0$, then agents of type θ_h will achieve the strictly greatest utility. Thus, the existence of an agricultural society requires the presence of both type-θ_w agents and type-θ_f agents working in unison. The warriors are required to drive down the utility of hunter-gatherers, while the farmers are the producers of agricultural output. Warriors' consumption is provided by the farmers through the tax t.

For any equilibrium to include a mix of types θ_w and θ_f, the two types must have the same utility. That is,

$$u_w(\boldsymbol{\mu}) = u_f(\boldsymbol{\mu}) \tag{1.5}$$
$$t\frac{\mu_f}{\mu_w} = \ell - t \tag{1.6}$$
$$\frac{\mu_w}{\mu_f} = \frac{t}{\ell - t}. \tag{1.7}$$

In an agricultural equilibrium there are no type-θ_h agents, therefore $\mu_h = 0$. Then, in any agricultural equilibrium, $\mu_f = 1 - \mu_w$. Substituting this equation into Equation (1.7) yields

$$\frac{\mu_w}{1 - \mu_w} = \frac{t}{\ell - t} \tag{1.8}$$
$$\mu_w = \frac{t}{\ell}, \tag{1.9}$$

and

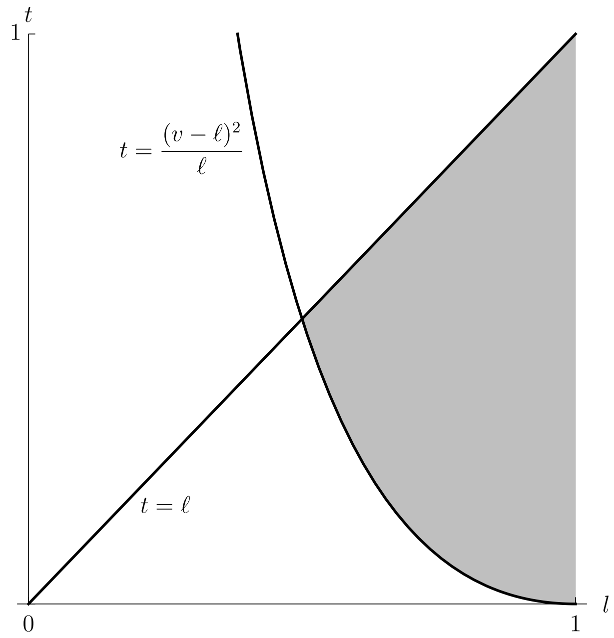
$$\mu_f = 1 - \frac{t}{\ell} \tag{1.10}$$
$$\mu_f = \frac{\ell - t}{\ell}. \tag{1.11}$$

The existence of an agricultural equilibrium will also require that the utility of type-θ_h agents is less than that of the other two types, implying

$$u_f(\boldsymbol{\mu}) \geq u_h(\boldsymbol{\mu}) \tag{1.12}$$
$$\ell - t \geq v\big(1 - \phi(\mu_w)\big) \tag{1.13}$$
$$v\phi(\mu_w) - t \geq v - \ell. \tag{1.14}$$

We summarise this discussion as follows.

Figure 1.1: For $v = 1$ and $\phi(\mu_w) = (\mu_w)^{\frac{1}{2}}$, the shaded region depicts the (ℓ, t) pairs for which an agricultural equilibrium exists



Proposition 1.2. *Given v, ℓ, and $\phi(\cdot)$, an agricultural equilibrium exists if and only if there exists $t \in [0, \ell]$ such that*

$$v\phi\left(\frac{t}{\ell}\right) - t \geq v - \ell. \tag{1.15}$$

$\boldsymbol{\mu} = \left(0, \frac{\ell - t}{\ell}, \frac{t}{\ell}\right)$ *is an equilibrium for all $t \in [0, \ell]$ satisfying Equation* (1.15).

The RHS of Equation (1.15) measures the premium that type-θ_h agents receive from hunting and gathering compared to the value that type-θ_f agents receive from farming. The LHS is the difference between the cost that type-θ_w agents inflict on type-θ_f agents, $v\phi\left(\frac{t}{\ell}\right)$, and the price paid by type-θ_f agents to support type-θ_w agents, t. The existence of equilibrium is dependent on the technology of violence, and there will be some functions ϕ for which no set of parameters will allow for an agricultural equilibrium. For instance, Figure 1.1 shows the set of parameter values on which the function $\phi(\mu_w) = (\mu_w)^{\frac{1}{2}}$ permits an agricultural equilibrium, but the function $\phi(\mu_w) = \mu_w$ will not permit an equilibrium for any set of parameter values.

In addition, some parameter values may permit all three types to co-exist simultaneously. For a mixed equilibrium to occur, it must be the case that all types receive the same expected utility, implying that Equation (1.7) is true and Constraint (1.14) holds with equality.

1.3.1 Efficiency

We measure the efficiency of the economy according to the aggregate utility, defined $U(\boldsymbol{\mu}) = \int_{i\in A} u_i di$. This is

$$U(\boldsymbol{\mu}) = \int_{i\in A} u_i di \tag{1.16}$$

$$= \mu_h v\big(1 - \phi(\mu_w)\big) + \mu_f(\ell - t) + \mu_w t\frac{\mu_f}{\mu_w} \tag{1.17}$$

$$= \mu_h v + \mu_f \ell - v\mu_h\phi(\mu_w). \tag{1.18}$$

The first two terms of Equation (1.18) measure the productive output of the economy, where all production is carried out by type-θ_h and type-θ_f agents. The final term measures the cost of violence between hunter-gatherers and warriors, a cost which is increasing in both μ_h and μ_w.

Assumption 1.1 guarantees the following proposition.

Proposition 1.3. *The hunter-gatherer equilibrium maximises aggregate utility.*

The proof is straightforward. Given Assumption 1.1, the vector $\boldsymbol{\mu} = (1, 0, 0)$ maximises $U(\boldsymbol{\mu})$, as given in Equation (1.18).

For any fixed parameterisation of v, ℓ, and $\phi(\cdot)$, the agricultural equilibrium that obtains the greatest aggregate utility is the equilibrium that minimises t. From Equation (1.18), in any agricultural equlibrium $U(\boldsymbol{\mu}) = \mu_f\ell$. Then, aggregate utility is increasing in μ_f, which from Equation (1.11) is $\mu_f = \frac{\ell - t}{\ell}$. Thus, a minimal value of t will maximise μ_f, in turn maximising $U(\boldsymbol{\mu})$.

As the tax t decreases, farmers keep a greater share of their yield, and they are more content with farming versus hunting and gathering. Then, fewer warriors are required to sufficiently threaten hunter-gatherers to entice them to farm. This results in a greater share of farmers relative to warriors, and the total yield shared between farmers and warriors is higher. In Figure 1.1, the agricultural equilibrium that maximises aggregate utility for each ℓ lies on the bottom boundary of the equilibrium region described by the curve $t = \frac{(v-\ell)^2}{\ell}$.

1.4 Warrior Elite

While we have demonstrated the potential for an agricultural equilibrium to exist through Proposition 1.2, Proposition 1.3 ensures that such an equilibrium would never be preferred over a hunter-gatherer equilibrium by any of the agents in A. Thus, an agricultural equilibrium represents a coordination failure for these agents. Given that, prior to transition, Neolithic societies were in a hunter-gatherer equilibrium, we intend to develop a potential shock that may have caused rational agents to depart from such an equilibrium.

We propose to amend the model by making the technology of violence, ϕ, an exclusive technology. That is, agents of type θ_f and θ_h will not have the ability to transition to type θ_w. Suppose, for instance, that a small group of individuals amass sufficient tools or weaponry to allow them dominance over the hunter-gatherers and farmers that surround them. Being in possession of the means of violence, this elite group of warriors would have the ability to restrict access to those whom they select. They would, in effect, have first mover advantage, as any attempts by non-warriors to stockpile arms independently would result in a swift violent rebuttal.

We define the *warrior elite* game as follows:

Stage 1: A single agent, $e \in A$, has access to the technology of violence. Agent e may choose to be type θ_w, in which case he will choose a mass of allies, $\mu_w \in (0, 1]$, and the tax, t. Agent e will select the pair (μ_w, t) that maximises his utility after Stage 2. Alternatively, e may choose to be type θ_h, in which case we set $\mu_w = 0$ and $t = 0$.

Stage 2: The non-type-θ_w agents, representing a mass of $1 - \mu_w$ in A, choose their type from the set $\{\theta_h, \theta_f\}$. If they are indifferent between types, they will choose type θ_f.

We solve the warrior elite game through backward induction. Let the set of non-type-θ_w agents be $A_{-w} = A \setminus A_w$. Any agent $i \in A_{-w}$ who chooses θ_h will receive payoff $u_h(\boldsymbol{\mu}) = v\big(1 - \phi(\mu_w)\big)$. Alternatively, any $i \in A_{-w}$ who chooses θ_f will receive payoff $u_f(\boldsymbol{\mu}) = \ell - t$. Then, Constraint (1.14) determines whether θ_h or θ_f generates a greater payoff for any $i \in A_{-w}$. We summarise the decision rule for any $i \in A_{-w}$ as follows:

$$\theta_i = \begin{cases} \theta_h, & \text{if } v\phi(\mu_w) - t < v - \ell, \text{ and} \\ \theta_f, & \text{otherwise.} \end{cases} \tag{1.19}$$

After Stage 2, if the agents in A_{-w} choose θ_h, then $u_w(\boldsymbol{\mu})$ will be zero. Thus, e may only derive positive utility when the agents in A_{-w} choose θ_f. Agent e's utility in this case is

$$u_w(\boldsymbol{\mu}) = t\frac{\mu_f}{\mu_w} \tag{1.20}$$

$$= t\frac{1 - \mu_w}{\mu_w}. \tag{1.21}$$

We may then solve for agent e's optimal choice of (μ_w, t) in Stage 1, using the following optimisation problem:

$$\begin{aligned} (\tilde{\mu}_w, \tilde{t}) = \ & \underset{\mu_w, t}{\arg\max} \quad t\frac{1 - \mu_w}{\mu_w} \\ & \text{subject to} \ \ v\phi(\mu_w) - t \geq v - \ell. \end{aligned} \tag{1.22}$$

If agent e's payoff from choosing $\left(\tilde{\mu}_w, \tilde{t}\right)$ is greater than v, then e will choose to be type θ_w and initiate an agricultural equilibrium. Otherwise, e will choose to be type θ_h and there will be a hunter-gatherer equilibrium. Agent e's decision rule is as follows:

$$\theta_e = \begin{cases} \theta_w, \text{ if } \tilde{t}\dfrac{1-\tilde{\mu}_w}{\tilde{\mu}_w} > v, \text{ and} \\ \theta_h, \text{ otherwise.} \end{cases} \tag{1.23}$$

Under the warrior elite game, the proof of Proposition 1.3 still holds: the hunter-gatherer equilibrium is always the efficient equilibrium. In contrast to our basic model, however, under the warrior elite game the yield from agriculture is not shared evenly among all participants in the agricultural society. Because of their fighting dominance, the type-θ_w agents may claim a disproportionate share of the agricultural yield for themselves. As long as the elite group of warriors can ensure themselves a distribution of the yield that exceeds their utility from hunting and gathering, they will force the entire society into an equilibrium that is worse for society from an aggregate utility standpoint.

The optimal choice of μ_w, which determines whether a warrior elite may form, is affected by two factors. As μ_w increases, the threat to type-θ_h agents rises, meaning that hunting and gathering becomes less rewarding relative to farming. This, in turn, allows e to raise t while still ensuring that any agent $i \in A_{-w}$ will choose farming, and the larger value of t will increase the amount of food shared amongst the type-θ_w agents. However, because the type-θ_w agents share their allotment of food equally, as μ_w increases each individual type-θ_w agent will receive a smaller portion of the agricultural yield.

In Section 1.5.2 we will examine in more detail the technologies of violence that allow e to profitably establish a warrior elite.

1.4.1 Dynamic Transition

The warrior elite game is presented as a static game; however, it may easily be contextualised as a dynamic game to describe a transition from a hunter-gatherer equilibrium to an agricultural equilibrium. Suppose that, once a warrior elite forms, the transition to an agricultural equilibrium is not instantaneous. Instead, we suppose that, once the warrior elite is established, type-θ_h agents transition to type θ_f at a rate proportional to the mass of type-θ_h agents.

Suppose that there exists an agricultural equilibrium in the static warrior elite game. Consider an intertemporal process where τ denotes the time elapsed from an initial starting point. Let h_τ, f_τ, and w_τ be the masses of types-θ_h, θ_f, and θ_w agents, respectively, at times $\tau \geq 0$. At time $\tau = 0$, assume that e has the option to form a warrior elite of size μ_w^*, where (μ_w^*, t^*) is the solution to Equation (1.22). We assume that the size of the warrior elite remains constant once it forms, and so $w_\tau = \mu_w^*$ for all $\tau \geq 0$.

At time $\tau = 0$, if a warrior elite forms then the state vector is $\boldsymbol{\mu}_0 = (1 - \mu_w^*, 0, \mu_w^*)$.

Assume that type-θ_h agents will transition to type θ_f at a rate of

$$\mathrm{d}h_\tau = -\lambda h_\tau \mathrm{d}\tau, \tag{1.24}$$

where $\lambda \in (0,1)$. This would imply that the rate at which type-θ_f agents appear is

$$\begin{aligned} \mathrm{d}f_\tau &= \lambda h_\tau \mathrm{d}\tau & (1.25)\\ &= \lambda(1-\mu_w^*-f_\tau)\,\mathrm{d}\tau. & (1.26) \end{aligned}$$

Equations (1.24) and (1.26) define the state vector $\boldsymbol{\mu}_\tau$ for all $\tau \geq 0$.

Assume that agent e discounts his future utility at a continuous discount factor $\delta \in [0,1]$. Then e's discounted lifetime utility, starting at time $\tau = 0$, is

$$U_e = \int_0^\infty u_e(\boldsymbol{\mu}_\tau)\,\mathrm{e}^{-\delta\tau}\mathrm{d}\tau. \tag{1.27}$$

At time $\tau = 0$, e will choose between creating a warrior elite or maintaining the status quo by remaining in a hunter-gatherer equilibrium. Agent e will choose to form a warrior elite if his discounted lifetime utility is greater under this path.

Define $\boldsymbol{\mu}_{\mu_w^*} = (0, 1-\mu_w^*, \mu_w^*)$. We propose the following.

Proposition 1.4. *For all $\delta \in [0,1]$, if*

$$\lambda > \frac{\delta^2 v}{u_w\left(\boldsymbol{\mu}_{\mu_w^*}\right) - \delta v}, \tag{1.28}$$

then agent e will create a warrior elite. Otherwise agent e will maintain the status quo.

Proposition 1.4 shows that, so long as there is an agricultural equilibrium in the warrior elite game, there are parameterisations of the dynamic transition that would entice e to form a warrior elite.

1.5 Comparative Statics

A key element of both of the models presented in this chapter that drives the feasibility of an agricultural equilibrium is ϕ, the technology of violence. The function ϕ measures how effectively a group of warriors can threaten and subjugate hunter-gatherers, driving down their survival probabilities and causing them to choose farming. As warriors get more efficient, a smaller group will be required, allowing more agents to work as farmers, increasing the total agricultural yield.

1.5.1 Base Model

In the base model, the technology of violence determines whether an agricultural equilibrium exists.

Proposition 1.5. *Fix parameters v and ℓ.*

1. *Suppose that technology of violence ϕ_g permits an agricultural equilibrium. Then, for any ϕ_ℓ such that $\phi_\ell(x) \geq \phi_g(x)\, \forall\, x \in [0,1]$, ϕ_ℓ permits an agricultural equilibrium.*

2. *Suppose that technology of violence ϕ_g does not permit an agricultural equilibrium. Then, for any ϕ_ℓ such that $\phi_\ell(x) \leq \phi_g(x)\, \forall\, x \in [0,1]$, ϕ_ℓ will not permit an agricultural equilibrium.*

Proposition 1.5 describes how we may classify technologies of violence according to a partial ordering, and how we may use this ordering to determine equilibrium existence. The first part of the proposition stipulates that, given a technology of violence ϕ_f that permits an agricultural equilibrium, any technology of violence that is at least as effective for all μ_w will also permit an agricultural equilibrium. The second part of the proposition states the reverse: given a technology of violence that does not permit an agricultural equilibrium, any technology of violence that is weakly less effective for all μ_w will also not permit an agricultural equilibrium.

Example 1.1. *As an example, consider the family of power functions $\phi(x) = x^y$ for some $y \in \mathbb{R}^+$. As a consequence of Proposition 1.5, we may construct well-defined intervals for which $\phi(x)$ will and will not permit an agricultural equilibrium, as long as there is at least one power function that does permit an agricultural equilibrium.*

Suppose there exists some $y \in \mathbb{R}^+$ for which $\phi(x) = x^y$ permits an equilibrium. Then, set

$$L = \limsup_{y \in \mathbb{R}^+},\ \textit{such that } \phi(x) = x^y \textit{ permits an equilbrium, and} \tag{1.29}$$

$$U = \liminf_{y \in \mathbb{R}^+},\ \textit{such that } \phi(x) = x^y \textit{ does not permit an equilbrium.} \tag{1.30}$$

Then, for all $a, b \in \mathbb{R}^+$ such that $a < L \leq U < b$, $\phi(x) = x^a$ permits an agricultural equilibrium, and $\phi(x) = x^b$ does not permit an agricultural equilibrium.

1.5.2 Warrior Elite

In the warrior elite game, an agricultural equilibrium will occur if there is a (μ_w, t) pair that yields $u_w(\boldsymbol{\mu}) > v$. A type-$\theta_w$ agent's utility in this game is highly dependent on ϕ, particularly on the steepness of ϕ on small values. Recall that each type-θ_w agent's share of the tax revenue is $t\frac{1-\mu_w}{\mu_w}$. Thus, as μ_w decreases, each agent's share will increase in two ways: the numerator increases as the number of type-θ_f agents grows, increasing yield, and the denominator shrinks as there are fewer type-θ_w agents to share the revenue.

Figure 1.2: Type-θ_w utility for different ϕ. $v = 1$, $\ell = 0.7$

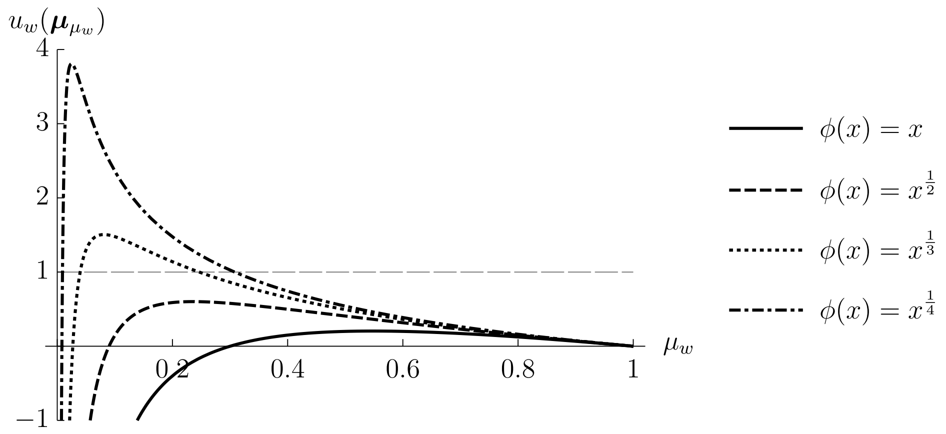


Figure 1.2 shows ϕ as a family of decreasing power functions. Each curve presents the utility of an individual type-θ_w agent, given the warrior elite is of mass μ_w. The tax, t, is computed implicitly to maximise $\phi(\mu_w)$. The peak of each curve occurs where the mass of the warrior elite maximises individual utilty for type-θ_w agents, which is the value for μ_w that agent e would select. For the first two curves, $\phi(x) = x$ and $\phi(x) = x^{\frac{1}{2}}$, the maximum value for $u_w\big(\boldsymbol{\mu}_{\mu_w}\big)$ is less than v. Agent e cannot profitably deviate from the hunter-gatherer equilibrium to form a warrior elite, as his utility post-deviation would fall.

Under the third curve, $\phi(x) = x^{\frac{1}{3}}$, a group of type-$\theta_w$ agents is more efficient at threatening type-θ_h agents than a group of the same mass would be under the first two curves. A relatively smaller group is able to sufficiently threaten the type-θ_h agents, resulting in a higher ratio of type-θ_f agents to type-θ_w agents. The peak of this curve is above $v = 1$, in Figure 1.2, which shows the level where agent e may profitably deviate from the hunter-gatherer equilibrium to form a warrior elite. Finally, the curve $\phi(x) = x^{\frac{1}{4}}$ shows that as the ϕ function increases in efficiency, a warrior elite that is decreasing in size may obtain increasingly greater utility.

1.6 Discussion

In this section, we contrast our model with the previous economic literature on the Neolithic Revolution. We draw predominantly from Weisdorf (2005), who summarises this literature into the general model reproduced in Figure 1.3.

At the time of the Neolithic Revolution, human societies choose between two forms of production: hunting and gathering, or agriculture. Agriculture provides a constant marginal productivity to each agent, whereas the marginal productivity from hunting and gathering declines with the size of the labour force. Prior to the Neolithic Revolution, relatively few human societies faced any forms of population pressure, and thus we assume the size of the labour force was at point L_1. Rational agents would choose hunting and gathering as a means of production.

Figure 1.3: The standardized model (Weisdorf 2005, Figure 1)

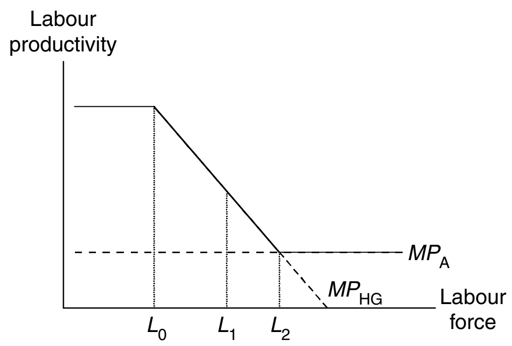
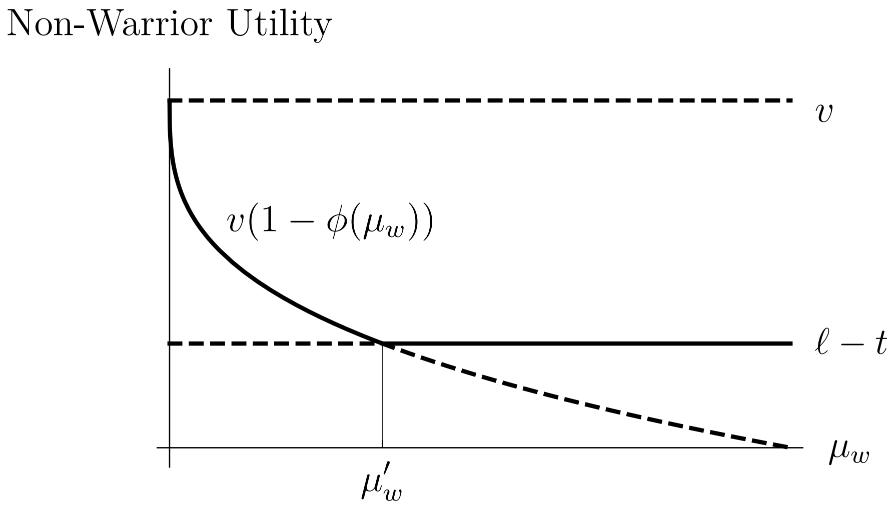


Figure 1.4: Non-warrior utility as a function of the mass of warriors



Generally, existing economic models postulate some form of shock to the forces at play in Figure 1.3. For instance, Smith (1975) hypothesises that overhunting led to a decline in the marginal productivity from hunting and gathering. This would be reflected in sufficiently large downward shift in the MP_{HG} curve such that, at the point L_1, MP_{A} exceeds MP_{HG}. Alternatively, Locay (1989) suggests that population growth may have preceded the Neolithic Revolution. A shock to the labour force to a level to the right of L_0 would sufficiently drive down the marginal productivity from hunting and gathering to make agricultural production a rational decision.

By contrast, Assumption 1.1 stipulates that in our model marginal productivity from hunting and gathering always exceeds the marginal productivity from agriculture in our model, as represented by the horizontal lines at v and $\ell - t$ in Figure 1.4. However, the technology of violence imposes a cost on all hunter-gatherers, which is a function of the mass of warriors in society, and the hunters' utility is the sum of their productive output less the cost of violence. While marginal productivity is constant, marginal utility for non-warriors is decreasing as the number of type-θ_w agents grows, as described by the line labeled $v\big(1-\phi(\mu_w)\big)$. When μ_w is less than the threshold μ_w', a rational non-warrior will choose hunting and gathering. If μ_w is greater than μ_w', then the threat of violence will be sufficient to cause non-warriors to choose agriculture.

1.7 Conclusion

We have contributed a new mechanism with which to interpret and potentially explain the Neolithic Revolution. While the anthropological literature on this period is expansive and increasing, there are relatively few economic contributions, exploring specifically the reward structures in place for humans during this period and the strategic incentives for humans to give up their hunting and gathering lifestyles in exchange for agricultural lifestyles.

In our model, hunting and gathering always provides a greater yield than agriculture. However, agents may specialise in violence, which allows them to threaten the survival of other agents. Agents who specialise in violence form a class of warriors, who threaten any agents who choose hunting and gathering as their activity. This threat reduces the expected utility from hunting and gathering, which may lead agents to select farming as an activity instead. The farmers then share the food they produce with the warriors. When an agricultural equilibrium exists, it will always result in a lower utility for all participants than a hunting and gathering equilibrium would.

We develop an additional restriction, where warriors form an elite group that all other agents are restricted from entering. This restriction allows for considerable inequality in the agricultural outcome, with the warriors receiving a greater share of production per agent than the farmers. While aggregate utility remains lower in an agricultural equilibrium compared with a hunter-gatherer equilibrium, the warrior elite will achieve utilities that are greater in an agricultural equilibrium, at the expense of farmers whose utilities are lesser. Thus, the warrior elite would have incentive to trigger a transition to an agricultural equilibrium from a starting point of a hunter-gatherer equilibrium. We model this as a static game, and also describe a dynamic process through which the transition may occur.

Chapter 2

Cournot Supply Chains

2.1 Introduction

Following a large catch, a fishing-boat captain may face an interesting consideration. Selling the entire catch to a single distributor could depress the price that the distributor is willing to pay, as he may not have a sufficiently large set of buyers to offload the supply to. Alternatively, the captain may choose to sell parts of the catch to multiple distributors, where each distributor would potentially pay a greater price per unit (Jensen 2007, provides evidence that fishermen will engage in this sort of behaviour). The distributors would then face a similar consideration when allocating the fish to their own buyers, to sell the fish in a single market or distribute the goods more widely. We replicate this phenomenon in a network setting, where the price that any buyer is willing to pay is a function of the quantity that they receive; sellers must refrain from overwhelming their buyers with supply, while simultaneously competing for market share against other sellers.

In the preponderance of economic intermediation models, intermediaries compete in prices. In such models, goods tend to flow as a single mass, with all goods selecting the cheapest path and small price fluctuations potentially causing large shifts in equilibria to new paths. These dynamics would not describe the small-scale fisheries industry, and we believe that a new model is warranted. For many resources, local prices may vary wildly according to the current supply, and a quantity-driven model may provide insight into a variety of commodity markets.

An historical example is the earliest oil markets in Western Pennsylvania beginning in the late 1850s (Tarbell 1904). There was a clear chain from source to market: first the drillers sold their oil to teamsters, who would sell their oil to refiners, who finally sold their oil to shippers in New York, who would send the oil overseas. The Standard Oil Company attained significant market power by consolidating all of the refiners, thus reducing the refining step to a single entity. The drillers remained in near perfect competition, and collectively they oversupplied raw oil to such an extent that Standard Oil was able to purchase their entire demanded quantity of oil at very low prices. In fact, at times oversupply was so extreme that drillers were forced to pump their excess oil back into the ground.

We assume that there is a single source of a resource, and a single market that demands this good. Between these lie a multipartite network of intermediaries, each extracting rent from buying the good and then selling the good at higher prices to intermediaries nearer to the market. In each tier, intermediaries must simultaneously decide how to allocate their quantity of resource to all of the buyers to whom they have connections. As profit maximisers, intermediaries seek to sell their quantity of the good at the highest possible price. However, the price a buyer will pay is determined by the joint demand function of all buyers in a tier. The price paid for the good by any intermediary decreases as their allocation of the good increases. The demand function is recursive, being determined by the revenues that the intermediaries will obtain when they sell the good to intermediaries in the next tier.

When two selling intermediaries are linked to the same buyer, their allocation decisions with respect to that buyer are substitutes. An increase in the quantity of the good provided from one seller to the buyer will increase the buyer's price, and thus reduce the marginal benefit that the other seller receives.

Our primary contribution is to develop a full analytical solution to the model. As the network is multipartite, the flow of the good can be broken down into a discrete series of steps between each pair of connected tiers in the network. Beginning at the final tier, intermediaries sell all of their good to the market, and marginal revenue is determined in the manner of standard Cournot competition, driven by the quantity that each intermediary has available to sell. When the intermediaries in the final tier buy the good from the second-from-last tier, the final tier intermediaries will pay a price equal to their marginal revenue per unit of the good received. This creates a demand schedule for the intermediaries in the second-from-last tier, who jointly allocate the good to their buyers in a manner that maximises their individual revenues. The marginal revenues received by each intermediary in the second-from-last tier create the demand schedule used by intermediaries in the third-from-last tier. Recursing through the network, we determine the optimal quantities that each intermediary in the initial tier will obtain from the source, which determines the flow of good through the entire network.

When buying the good, each intermediary functions locally as a market with an oligopoly of sellers, with each connected seller engaging in a Cournot competition. Each seller is differentiated by their set of out-links, which allows them to select in which markets (buyers) to participate. In each market, the sellers are price setters, with their choice of quantity affecting the market price. As such, the profit maximising decision for any seller maximises marginal revenue, not price. We examine the local decision making processes within each tier, and develop an approximation to determine the amount of the good that will be allocated to any new link.

Using the analytical solution, we determine the networks that generate the least and greatest flow of the good to the final market. Using these networks we find closed-form bounds on the maximal and minimal quantity of good traded based upon the number of tiers and the number of intermediaries in each tier in a network. We develop a measure, the price of incompleteness, that evaluates networks according to the consumer surplus they generate in comparison to a dense network.

Corbett and Karmarkar (2001) develop a model of quantity competition in a supply chain. Intermediaries are separated into tiers, and in each tier agents simultaneously choose quantities that determine the price paid by agents in the next tier. In contrast to our model, there are no linking structures, as the pool of resource provided by intermediaries in one tier is available to all intermediaries in the next tier, and thus all markets must be complete. Bimpikis, Candogan et al. (2019) look at the resilience of multi-tiered supply chains under adverse shocks. Like Corbett and Karmarkar, they assume that markets are complete between each tier. Nagurney et al. (2002) describe a two-step chain of quantity competition, where manufacturers produce goods for retailers, who then sell the goods to consumers. Each stage is complete, but links have heterogeneous costs associated with selling the good. Adida and DeMiguel (2011) describe a single-step supply chain where a set of manufacturers provide multiple products to a set of retailers with stochastic demand. They allow for asymmetries in demand, which lead to heterogeneous product provision. Owing to the complexity of their models, both Adida and DeMiguel (2011) and Nagurney et al. (2002) use numerical methods to approximate equilibrium. Alternatively, we develop an analytical solution to our model, which provides greater scope for analysis.

Manea (2018) analyses a directed acyclic network of intermediaries buying and reselling an indivisible good. Intermediaries bargain over price, and Manea develops an algorithm for determining which intermediaries hold bargaining power. In Manea (2019), the good is a duplicable information good that diffuses through an undirected network of intermediaries. They find a method of partitioning the network into equivalence classes that determines which intermediaries are essential. Both models demonstrate how the nature of transactions determines which intermediaries benefit most in a network setting. Our network is most similar to Manea's (2018), but we find that when intermediaries compete in quantities, as opposed to prices, profits and the flow of goods are more evenly distributed amongst intermediaries.

Two papers draw explicitly upon applications of Cournot competition in networks. Bimpikis, Ehsani et al. (2019) propose a one-step game, with a tier of firms selling to a tier of heterogeneous markets. In structure, our game is similar, particularly with regard to the strategic decision making between competing firms. Our contribution is to develop a game on a multi-tiered intermediation chain, where decision making is sequential and intermediaries must consider the actions of others above and below them in the network. Nava (2015) proposes a model where agents in an undirected network receive heterogeneous endowments of a good. Then, a series of transfers occurs, where the good is redistributed between the nodes and the price paid by any node is determined by that node's demand and the total amount of good they receive.

That being central within a network—that is, being an agent who bridges other agents—brings value in a trading setting is a well-explored concept. In an undirected network, Choi et al. (2017) propose a model where two agents can trade to create a surplus, and all other agents post the price they would charge to allow trade to flow through their node. They find that it is essential agents, those who lie on every path between the traders, that are able to demand an intermediation rent, and they perform laboratory experiments that confirm human players will employ the expected strategies.

We proceed as follows: Section 2.2 defines a network and outlines the game played by all intermediaries. Section 2.3 begins with a worked example of equilibrium on a sample network, before presenting an analytical solution for the unique equilibrium on any network. Section 2.4 analyses the decision-making process for intermediaries in greater depth and presents an approximation for finding the profit-maximising new out-link for any intermediary. Section 2.5 evaluates the effect of varying network structures on consumer surplus. Section 2.6 concludes. All proofs are found in Appendix B.2.

2.2 Model

2.2.1 Network

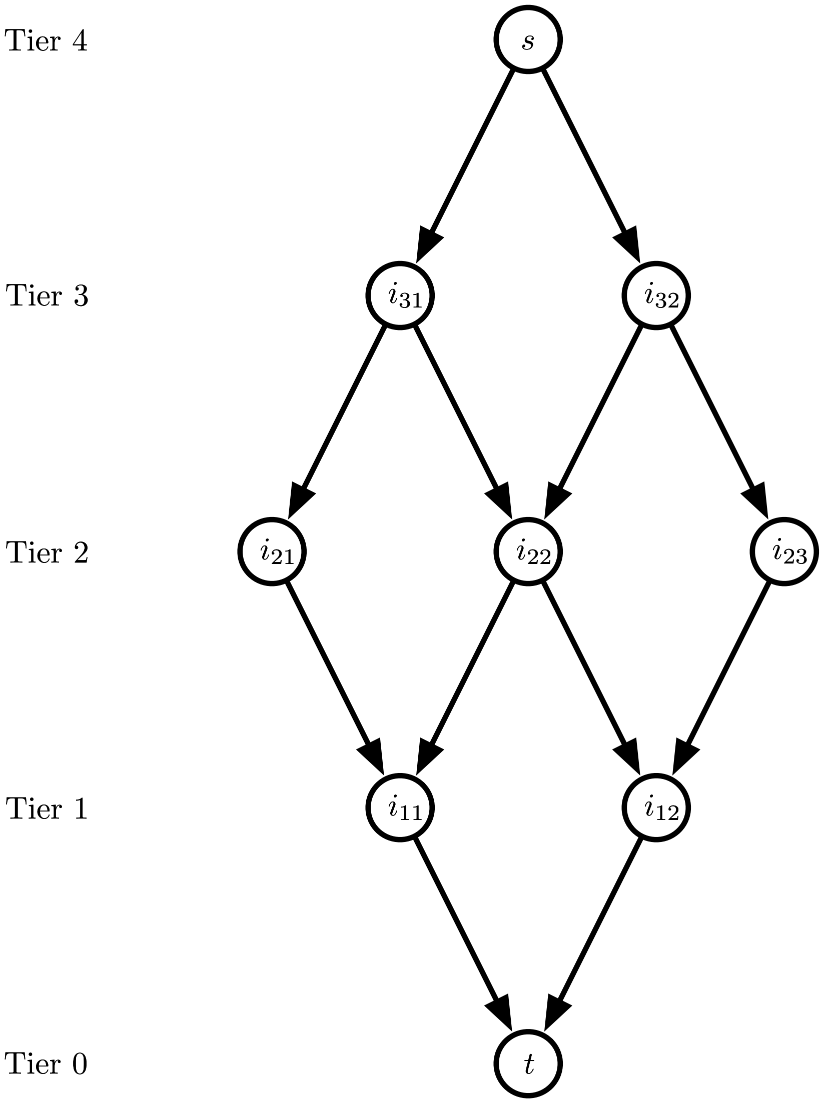
There are $n+2$ nodes in a directed multipartite network, each representing an individual or firm. These are a source node, s, a target (or sink) node, t, and n intermediaries in the set I. The network is arranged into $m+2$ *tiers*. Tier 0 contains only node t, tier $m+1$ contains only node s, and each intermediary $i \in I$ exists in one of tiers $\{1, \ldots, m\}$. The set of nodes in tier x is $I_x \subset I$. A partition of the nodes in i into the sets $\{I_1, \ldots, I_m\}$ is denoted $\mathcal{P}(I)$.

The nodes in the network are connected by a set of edges, E. Each edge is directed, connecting a tail in tier x to a head in tier $x-1$. The existence of an edge indicates that the tail may sell goods to the head. The set of edges that runs from tier x to tier $x-1$ is $E_{x,x-1} \subset E$.

A network g is a collection of the two sets $I \cup s \cup t$ and E. We assume that an edge exists from s to each node in tier m, and from each node in tier 1 to t. The remaining network is described by the variable g_{ij} defined on each ordered pair ij such that $i, j \in I$. $g_{ij} = 1$ if the edge $e_{i,j}$ exists in g, and $g_{ij} = 0$ if it does not.

For any node $i \in I$, we denote any edge for which i is the head as an *in-link*, and any edge for which i is the tail as an *out-link*. The set of nodes from which i has an in-link is i's *in-neighbourhood*, defined $N_i^+(g) = \{j \in I \cup s \mid g_{ji} = 1\}$. Similarly, i's *out-neighbourhood* is the set $N_i^-(g) = \{j \in I \cup t \mid g_{ij} = 1\}$. We define the sets of i's in-links and out-links as $L_i^+(g) = \{ji \in E \mid g_{ji} = 1\}$ and $L_i^-(g) = \{ij \in E \mid g_{ij} = 1\}$, respectively. We define i's *in-degree* and *out-degree* as $d_i^+ = \left|N_i^+(g)\right|$ and $d_i^- = \left|N_i^-(g)\right|$, respectively. The members of i's in-neighbourhood may also be referred to as i's upstream neighbours, and similarly,

Figure 2.1: A network with $n = 8$ and $m = 3$



the members of i's out-neighbourhood are i's downstream neighbours. We define a path of length ℓ from i to j to be a sequence of distinct nodes, $i, i_1, i_2, \ldots, i_{\ell-1}, j$ such that $g_{ii_1} = g_{i_1 i_2} = \cdots = g_{i_{\ell-1}j} = 1$.

We assume that all nodes have at least one in-link and at least one out-link, so that each $i \in I$ lies on at least one path between s and t, and all paths between s and t are of length $m + 1$. Any network g belongs to the class of directed multipartite acyclic graphs. An example network is presented in Figure 2.1.

2.2.2 Competition Structure

There is a single renewable resource at node s. All nodes in tier m have equal access to this good. There is an infinite quantity of the good available, and it may be infinitely divided into any positive real quantities.

There is a single market for the good at node t. At the market, there is a representative consumer whose price is set according to an inverse demand function $p(q_t)$, where q_t is the total quantity of good available for purchase. We assume that the inverse demand function is linear, defined

$$p(q_t) = 1 - q_t. \tag{2.1}$$

At each node $i \in I$ there is an intermediary. Each intermediary takes two actions: first, he purchases the good from intermediaries in his in-neighbourhood; second, he allocates the good to intermediaries in his out-neighbourhood. Within each tier, all intermediaries act simultaneously, and the result of their actions governs the actions of the following tier. The first tier to act is tier m, followed by $m-1$, $m-2$, and proceeding sequentially until the intermediaries in tier 1 act. Intermediaries are profit maximisers, whose profit is derived from the ability to resell the good at a greater price than their own purchase price. We break down the two steps in the section below.

Action sequence for tier $x \in \{2, \ldots, m-1\}$

Step 1: Buying the Good

Assume that, for all tiers $y \in \{x+1, ...m\}$, all intermediaries in I_y have acted; that is, they have purchased the quantity of good they received and allocated a portion of the good to each of their out-neighbours.

Then, the quantity $q_{k,i}$ that describes the amount of good flowing along each link $e_{k,i} \in E_{x+1,x}$ is well defined. Each intermediary $i \in I_x$ receives the sum of good provided along each of his in-links, which is

$$q_i = \sum_{k \in N_i^+(g)} q_{k,i}. \tag{2.2}$$

The vector of quantities available to all intermediaries in tier x is $\boldsymbol{q}_x \in \mathbb{R}_+^{|I_x|}$.

When each intermediary is in possession of the good, they will allocate the good to each of their downstream neighbours, resulting in some revenue for each intermediary $i \in I_x$, $r_i(\boldsymbol{q}_x)$, which is a function of $\boldsymbol{q}_x$. All intermediaries pay a price for the good equal to their willingness-to-pay; that is, their marginal revenue from receiving additional quantity of the good:

$$p_i(\boldsymbol{q}_x) = \frac{\partial r_i(\boldsymbol{q}_x)}{\partial q_i}. \tag{2.3}$$

The vector-valued function that determines the prices paid by all intermediaries $i \in I_x$ is $p_x(\boldsymbol{q}_x) : \mathbb{R}_+^{|I_x|} \to \mathbb{R}_+^{|I_x|}$. This function describes the prices that each $i \in I_x$ will pay for each unit of good provided by an upstream neighbour in tier $x+1$.

Step 2: Selling the Good

Assume that there exists some vector $\boldsymbol{q}_x \in \mathbb{R}_+^{|I_x|}$.

Each intermediary $i \in I_x$ must allocate the good to their downstream neighbours. Intermediary i does so by choosing an amount $q_{i,j} \geq 0 \, \forall \, j \in N_i^-(g)$, and we set

$q_{i,j} = 0 \, \forall \, j \notin N_i^-(g)$. An intermediary may not sell more good than they receive, and thus i's strategy is subject to the constraint that

$$\sum_{j \in N_i^-(g)} q_{i,j} \leq q_i. \tag{2.4}$$

Intermediary i's action is the vector $(q_{i,j}) \in S_i \, \forall \, j \in N_i^-(g)$, where i's action space is the simplex $S_i \subset \mathbb{R}_+^{d_i^-}$ described by the constraint in Equation (2.4).

We assume that intermediaries in I_x may condition their strategies on the vector $\boldsymbol{q}_x$, but not on the individual actions of any intermediaries in tiers $x+1, \ldots, m$. A strategy for i is a function $s_i(\boldsymbol{q}_x) : \mathbb{R}_+^{|I_x|} \to S_i$ that describes i's allocation given any vector $\boldsymbol{q}_x$.

The set of strategies $s_i(\boldsymbol{q}_x) \, \forall \, i \in I_x$ will determine the quantity of good provided along each link $e \in E_{x,x-1}$. These quantities, in turn, will determine the quantities available to each $j \in I_{x-1}$, according to Equation (2.2), giving the vector $\boldsymbol{q}_{x-1}$. Step 1 describes how $\boldsymbol{q}_{x-1}$ determines the prices paid by each intermediary in tier $x-1$, $p_j(\boldsymbol{q}_{x-1}) \, \forall \, j \in I_{x-1}$.

Once all intermediaries in tier x have employed their strategies, the revenue received by each intermediary $i \in I_x$ is as follows:

$$r_i = \sum_{j \in N_i^-(g)} q_{i,j} p_j(\boldsymbol{q}_{x-1}). \tag{2.5}$$

Intermediaries are profit maximisers. The profit for any intermediary $i \in I_x$ is determined by the difference between his revenue and cost according to the following function

$$\pi_i(\boldsymbol{s}, g) = r_i - q_i p_i(\boldsymbol{q}_x), \tag{2.6}$$

where $\boldsymbol{s}$ is the vector of strategies $(s_i) \, \forall \, i \in I$, which determines the quantity of the good that is provided along each link. We make the simplifying assumption that there is no processing cost for each intermediary. Adding a fixed cost per unit reduces equilibrium prices by the amount of the fixed cost, which also has the effect of reducing equilibrium quantities and profits.

2.3 Equilibrium

The equilibrium concept employed is subgame-perfect-Nash equilibrium (SPE). Recall that we assume agents in tier x may only condition their strategies on the vector $\boldsymbol{q}_x$, and not the individual actions of any intermediaries in tiers $\{x+1, \ldots, m\}$. Then, for any tier x, any vector $\boldsymbol{q}_x \in \mathbb{R}_+^{|I_x|}$ describes a subgame where the intermediaries in I_x are active. In making their allocation decisions, the intermediaries in I_x determine $\boldsymbol{q}_{x-1}$, and thus select the next subgame in our model.

The final subgame is described by $\boldsymbol{q}_1$. As all intermediaries in tier 1 have a single out-link to the market (t), the intermediaries in I_1 engage in a standard Cournot competition, with the constraint that $q_{it} \leq q_i \; \forall \, i \in I_1$. The outcome of subgame $\boldsymbol{q}_1$ determines q_t, which subsequently determines p_t according to Equation (2.1).

As a condition of SPE, the decisions of each intermediary must induce a Nash equilibrium at every subgame. That is, in any subgame $\boldsymbol{q}_j$, each intermediary's strategy must be a best response to each other intermediary's strategy at that tier. Let $s_i(\boldsymbol{q}_j)$ be intermediary i's action in subgame $\boldsymbol{q}_x$. Let $\boldsymbol{s}_{-i}(\boldsymbol{q}_x)$ be the set of actions for all intermediaries in the set $\left(\cup_{y\in\{1,\ldots,x\}} I_y\right) \setminus i$. Then, we formally define an SPE $\boldsymbol{s}^*$ as the following: for all $x \in \{2, \ldots, m\}$ with $\ell = |I_x|$, for all $\boldsymbol{q}_x \in \mathbb{R}^{\ell}_{+}$, and for all $i \in \cup_{y\in\{1,\ldots,j\}} I_y$

$$\pi_x \left(s_i^* \left(\boldsymbol{q}_x\right), \boldsymbol{s}_{-i} \left(\boldsymbol{q}_x\right), g\right) \geq \pi_x \left(\tilde{s}, \boldsymbol{s}_{-i} \left(\boldsymbol{q}_j\right), g\right), \tag{2.7}$$

for all $\tilde{s} \in S_i$.

In Section 2.3.2 we will derive an explicit characterisation of the unique SPE that exists on any network. To clarify the model, however, we first derive the equilibrium of a simple network.

2.3.1 Equilibrium Example

Example 2.1. *Consider the network g_s in Figure 2.2, with $n = 4$ and $m = 2$.*

Tier 0: *The price at node t is*

$$p_t = 1 - q_t. \tag{2.8}$$

Tier 1: *Intermediaries i_{11} and i_{12} both sell to node t. The quantity available at t is*

$$q_t = q_{11,t} + q_{22,t}. \tag{2.9}$$

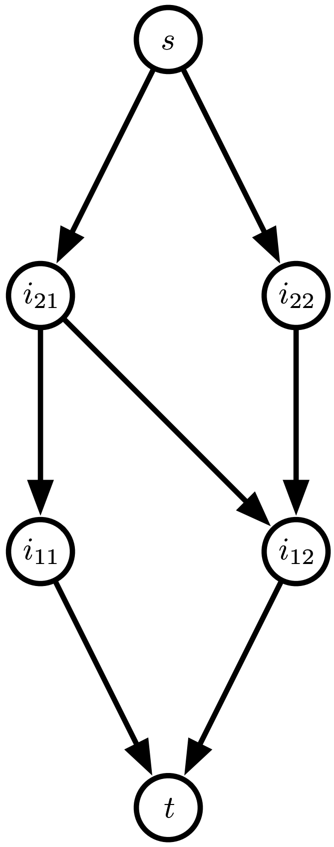
Both i_{11} and i_{12} sell their entire quantity of good to t, and therefore

$$q_{11,t} = q_{11} \tag{2.10}$$

and

$$q_{12,t} = q_{12}. \tag{2.11}$$

Figure 2.2: The network g_s



The revenue received by i_{11} as a function of q_{11} and q_{12} is

$$\begin{aligned}
r_{11}(q_{11}, q_{12}) &= p_t q_{11,t} && (2.12)\\
&= (1 - q_t)\, q_{11,t} && (2.13)\\
&= \left(1 - q_{11,t} - q_{12,t}\right) q_{11,t} && (2.14)\\
&= (1 - q_{11} - q_{12})\, q_{11} && (2.15)\\
&= q_{11} - q_{11}q_{12} - q_{11}^2. && (2.16)
\end{aligned}$$

Finally, the price paid by i_{11} is equal to his marginal revenue per quantity received, which is

$$\begin{aligned}
p_{11}(q_{11}, q_{12}) &= \tfrac{\partial r_{11}(q_{11}, q_{12})}{\partial q_{11}} && (2.17)\\
&= \tfrac{\partial}{\partial q_{11}}\left(q_{11} - q_{11}q_{12} - q_{11}^2\right) && (2.18)\\
&= 1 - 2q_{11} - q_{12}. && (2.19)
\end{aligned}$$

Similar calculations reveal that

$$p_{12}(q_{11}, q_{12}) = 1 - q_{11} - 2q_{12}. \qquad (2.20)$$

Tier 2: *The quantities of good arriving at* i_{11} *and* i_{12} *are*

$$q_{11} = q_{21,11} \tag{2.21}$$

and

$$q_{12} = q_{21,12} + q_{22,12}. \tag{2.22}$$

Intermediary i_{21} *may sell his quantity to* i_{11} *or* i_{12}*, and* i_{22} *may only sell his quantity to* i_{12}*. Therefore,*

$$q_{21,11} + q_{21,12} = q_{21}. \tag{2.23}$$

and

$$q_{22,12} = q_{22}. \tag{2.24}$$

Intermediary i_{21}*'s revenue is*

$$\begin{aligned}
r_{21}(q_{21,11}, q_{21,12}; q_{12}) &= p_{11}q_{21,11} + p_{12}q_{21,12} && (2.25)\\
&= (1 - 2q_{11} - q_{12})q_{21,11} + (1 - q_{11} - 2q_{12})\, q_{21,12} && (2.26)\\
&= \Big(1 - 2q_{21,11} - \big(q_{21,12} + q_{22,12}\big)\Big)\, q_{21,11} && (2.27)\\
&\qquad + \Big(1 - q_{21,11} - 2\left(q_{21,12} + q_{22,12}\right)\Big)\, q_{21,12} &&\\
&= \Big(1 - 2q_{21,11} - \big(q_{21,12} + q_{22}\big)\Big)\, q_{21,11} && (2.28)\\
&\qquad + \Big(1 - q_{21,11} - 2\left(q_{21,12} + q_{22}\right)\Big)\, q_{21,12}. &&
\end{aligned}$$

Intermediary i_{21} *selects* $q_{21,11}$ *and* $q_{21,12}$ *in such a manner to maximise his revenue. Intermediary* i_{21} *acts simultaneously with* i_{22}*. However, because* i_{22} *has only one out-link,* i_{21} *may take* i_{22}*'s actions as given. Intermediary* i_{21}*'s optimisation problem is then*

$$\begin{aligned}
(q_{21,11}, q_{21,12}) = \quad & \underset{a,\,b}{\arg\max} \quad \big(1 - 2a - (b + q_{22})\big)\, a + \big(1 - a - 2\,(b + q_{22})\big)\, b \\
& \text{subject to} \quad a + b \le q_{21}.
\end{aligned} \tag{2.29}$$

Solving this optimisation problem yields

$$q_{21,11} = \frac{1}{2}q_{21} + \frac{1}{4}q_{22} \tag{2.30}$$

and

$$q_{21,12} = \frac{1}{2}q_{21} - \frac{1}{4}q_{22}. \tag{2.31}$$

Substituting Equations (2.24), (2.30) *and* (2.31) *into Equation* (2.28) *yields* i_{21}*'s*

Table 2.1: Equilibrium quantities, profits, and prices for network g_s

Node	Quantity	Price	Profit
i_{21}	0.25	0	0.0972
i_{22}	0.1667	0	0.0556
i_{11}	0.1667	0.4167	0.0278
i_{12}	0.25	0.3333	0.0625
t	0.4167	0.5833	CS = 0.0868

revenue with respect to quantities q_{21} and q_{22}:

$$r_{21}(q_{21}, q_{22}) = q_{21} - \frac{3}{2}q_{21}^2 - \frac{3}{2}q_{21}q_{22} + \frac{1}{8}q_{22}^2. \tag{2.32}$$

Finally,

$$\begin{aligned} p_{21}(q_{21}, q_{22}) &= \tfrac{\partial r_{21}(q_{21}, q_{22})}{\partial q_{21}} && (2.33) \\ &= 1 - 3q_{21} - \frac{3}{2}q_{22}. && (2.34) \end{aligned}$$

And, by similar calculations,

$$p_{22}(q_{21}, q_{22}) = 1 - \frac{3}{2}q_{21} - \frac{15}{4}q_{22}. \tag{2.35}$$

By assumption, intermediaries in the top tier have access to the good at zero cost. To derive the quantity of good flowing through the network, we solve the system of equations obtained from setting the prices in Equations (2.34) *and* (2.35) *to zero.*

$$\begin{aligned} 1 - 3q_{21} - \frac{3}{2}q_{22} &= 0, \text{ and} && (2.36) \\ 1 - \frac{3}{2}q_{21} - \frac{15}{4}q_{22} &= 0. && (2.37) \end{aligned}$$

This yields the following quantities for i_{21} and i_{22},

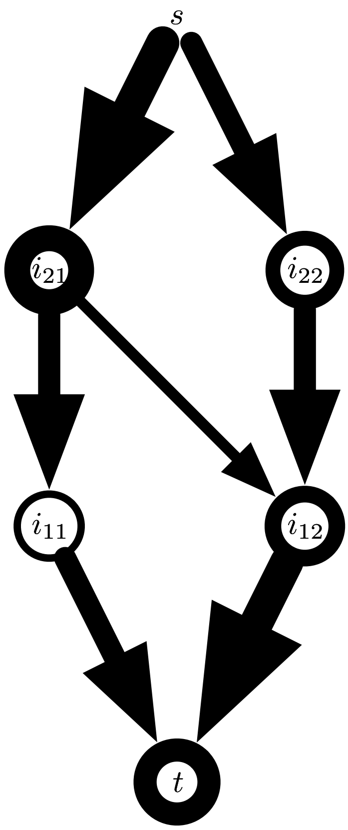
$$q_{21} = \frac{1}{4} \tag{2.38}$$

and

$$q_{22} = \frac{1}{6}. \tag{2.39}$$

The initial quantities chosen by i_{21} and i_{22} determine the quantity of goods flowing along each downstream link in the network g_s. Table 2.1 shows the equilibrium quantities, prices, and profits for the network g_s. Figure 2.3 graphically represents the equilibrium flow of good through the network; each edge is weighted by the amount of good flowing along the edge, and each node is weighted by the profit received by the intermediary at the node.

Figure 2.3: Equilibrium quantities and profits for the network g_s



2.3.2 Equilibrium Characterisation

In Section 2.3.1, we proceeded with the assumption that, in equilibrium, any intermediary will allocate their entire quantity of good to their downstream neighbours. While not a prerequisite for the model, the logic that supports this assumption is straightforward. The price paid by any intermediary is equal to their marginal revenue from additional units received. If it is optimal for an intermediary $i \in I$ to allocate less than his entire quantity of the good to his downstream neighbours, then i is not making use of his marginal units of the good and his marginal revenue is zero. But, then i's price must also be zero, and no upstream intermediary would be willing to sell any quantity of the good to i, which would mean that i's quantity of the good must be zero. We formally state this in the following proposition.

Proposition 2.1. *In any equilibrium $\boldsymbol{s}^*$ on a network g,*

$$\sum_{j \in N_i^+(g)} q_{i,j} = q_i \tag{2.40}$$

for all $i \in I$.

Equilibrium quantities are determined through backward induction, beginning in tier 1 and proceeding through the network until tier m. In tier 1, Proposition 2.1 ensures that all intermediaries sell their entire quantity of good to the single market. Then, for any quantity vector $\boldsymbol{q}_1$,

$$q_t = \sum_{i \in I_1} q_i. \tag{2.41}$$

Equation (2.1) determines p_t, and for each $i \in I_1$, i's marginal revenue from q_i is

$$\frac{\mathrm{d}r_i}{\mathrm{d}q_i} = \frac{\mathrm{d}}{\mathrm{d}q_i} p_t q_i \tag{2.42}$$

$$= p_t + q_i \frac{\partial p_t}{\partial q_i} \tag{2.43}$$

$$= p_t - q_i. \tag{2.44}$$

Calculating $\frac{\mathrm{d}r_i}{\mathrm{d}q_i}$ for each $i \in I_1$ will yield the unique vector-valued function $p_1(\boldsymbol{q_1})$.

When the intermediaries in tier 2 act, they have two key pieces of information: $\boldsymbol{q}_2$, the quantities of the good that each of the intermediaries in tier 2 holds, and $p_1(\boldsymbol{q_1})$, the function that determines the prices that each of the intermediaries in tier 2 will receive after making their simultaneous allocations. In equilibrium, these intermediaries will convert this information to a unique set of strategies satisfying the SPE condition in Equation (2.7). The process to determine each intermediary's action is threefold. First, we show that the intermediaries $i \in I_2$ are participating in a potential game, indicating that there is a potential function, $\Phi_2(\boldsymbol{q_2})$, whose unique global maximum is bijective to each individual intermediary's profit maximising actions (see Monderer and Shapley 1996). Second, we perform Karush-Kuhn-Tucker optimisation to find the actions for each intermediary in tier 2 that maximise $\Phi_2(\boldsymbol{q_2})$. Finally, we determine prices $p_2(\boldsymbol{q_2})$ by finding $\frac{\mathrm{d}r_i}{\mathrm{d}q_i}$ for each $i \in I_2$ at the optimal actions.

The following proposition describes this process algebraically.

Proposition 2.2. *Assume that the actions of all intermediaries describe an SPE. Consider tiers $x-1$ and x, where $|I_{x-1}| = \ell$ and $|I_x| = k$. Suppose there exists a matrix $\boldsymbol{X}_{x-1}$ such that $\boldsymbol{p}_{x-1} = \boldsymbol{1}_\ell - \boldsymbol{X}_{x-1}\boldsymbol{q}_{x-1}$. Then, there exist unique well-defined matrices $\boldsymbol{U}_x$, $\boldsymbol{D}_x$, and $\boldsymbol{F}_x$ such that*

$$\boldsymbol{p}_x = \boldsymbol{1}_k - \boldsymbol{X}_x \boldsymbol{q}_x, \tag{2.45}$$

where

$$\boldsymbol{X}_x = 2\left[\boldsymbol{U}_x \left(\boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^\mathsf{T} \circ \boldsymbol{F}_x\right)^{-1} \boldsymbol{U}_x^\mathsf{T}\right]^{-1}, \tag{2.46}$$

and $\circ$ denotes the Hadamard product.

Using Proposition 2.2, we may generate the function $p_2(\boldsymbol{q_2})$ given the inputs $p_1(\boldsymbol{q_1})$. This is the information that intermediaries in tier 3 will need, along with their quantity allocations $\boldsymbol{q}_3$, to generate their own actions. Repeated applications of Proposition 2.2 may then be used to generate the pricing functions $p_x(\boldsymbol{q}_x)$ for each $x \in \{1, \ldots, m\}$, which are defined by the matrices $\boldsymbol{X}_x$ for each tier $x \in \{1, \ldots, m\}$.

At this stage, pricing in the network is determined for any set of network flows, but the actual equilibrium strategy, $\boldsymbol{s}^*$, has yet to be determined. To calculate quantities requires a set of prices for the agents in the top tier, and we make use of the assumption that $\boldsymbol{p}_m = \mathbf{0}$. Then, rearranging Equation (2.45) for $\boldsymbol{q}_m$ yields

$$\boldsymbol{q}_m = \boldsymbol{X}_m^{-1}\mathbf{1}, \tag{2.47}$$

with $\mathbf{1}$ denoting the unit vector.

The proof of Proposition 2.2 (which is in Appendix B.2) includes the determination of each intermediary's individual action, which determines the quantity allocated on each link between tiers x and $x-1$, $\boldsymbol{q}_{x,x-1}$, and the quantity vector for tier $x-1$, $\boldsymbol{q}_{x-1}$, given the quantities $\boldsymbol{q}_x$, for any $x \in \{2,\dots,m\}$. These functions are presented in the following corollary.

Corollary 2.1. *Given $\boldsymbol{q}_x$, for any $x \in \{2,\dots,m\}$, the equilibrium quantity allocations along each link in $E_{x,x-1}$ are*

$$\boldsymbol{q}_{x,x-1} = \boldsymbol{A}_x^{\mathsf{T}}\boldsymbol{q}_x, \tag{2.48}$$

and the equilibrium quantity allocations to the intermediaries in tier $x-1$ are

$$\boldsymbol{q}_{x-1} = \boldsymbol{D}_x^{\mathsf{T}}\boldsymbol{A}_x^{\mathsf{T}}\boldsymbol{q}_x, \tag{2.49}$$

where

$$\boldsymbol{A}_x = \frac{1}{2}\boldsymbol{X}_x\boldsymbol{U}_x\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^{\mathsf{T}} \circ \boldsymbol{F}_x^{\mathsf{T}}\right)^{-1}. \tag{2.50}$$

Beginning with the quantities $\boldsymbol{q}_m$, repeated applications of Corollary 2.1 will determine the quantity flow of the good through the entire network.

The derivation of each of the matrices $\boldsymbol{D}_x$, $\boldsymbol{U}_x$, and $\boldsymbol{F}_x$ is left to Appendix B.1. $\boldsymbol{D}_x$, which we refer to as the downstream matrix, is a binary matrix that maps the set of edges $E_{x,x-1}$ to the intermediaries in I_{x-1}. $\boldsymbol{U}_x$, the upstream matrix, maps the intermediaries in I_x to the edges in $E_{x,x-1}$. Finally, $\boldsymbol{F}_x$, the potential adjustment, is used to convert the revenue from each edge to a potential game for the intermediaries in I_x. For each intermediary in I_x, $\boldsymbol{F}_x$ provides a weighted sum of the revenues according to whether or not edges share the same tails in I_x

2.4 Adding and Removing Links

Adding a link between intermediaries who are not already connected will create two new opportunities. For the seller, he has access to a new buyer of the good; allocating good to the new buyer will create a new revenue stream, and reducing quantity supplied to his existing buyers will increase the prices they pay. For the buyer, he has one more seller competing to sell him the good, reducing each individual seller's ability to influence his price. In this section, we address the question of which potential links will receive the greatest quantity of the good when added.

Finding such links achieves two purposes. First, sellers allocate their quantity of good in order to maximise their revenues, which is equivalent to equalising their marginal revenues along each outgoing link. Amongst all possible new out-links for a seller, the link that, when added, would carry the maximal quantity of good for any seller leaves a minimal amount of good allocated to existing links, which means that it must also be the new link that maximises marginal revenue and profit. Second, in Section 2.5, we discuss how consumer surplus for any network is an increasing function of the quantity of the good arriving at node t. While our analysis is limited to the local impact within a network of adding a new link, a social planner whose goal is to increase the total quantity of good flowing through the entire network would likely succeed by focussing on new links that would carry a maximal amount of good.

The allocation decision for an intermediary with a single out-link is straightforward: he will allocate his entire quantity of good to his single buyer. Otherwise, as revenue maximisers, intermediaries will distribute their good such that the marginal benefit they receive from the quantity of good provided to each link is equal. When intermediary i has an out-link to j, i's revenue along this link is $q_{ij}p_j(q_{ij}, \boldsymbol{q}_{-ij})$, where we abuse notation and allow $\boldsymbol{q}_{-ij}$ to be set of quantities provided along all links in the network excluding e_{ij}. Holding $\boldsymbol{q}_{-ij}$ fixed, i's marginal revenue with respect to q_{ij} is

$$\frac{\mathrm{d}}{\mathrm{d}q_{ij}} q_{ij}p_j(q_{ij}, \boldsymbol{q}_{-ij}) = p_j + q_{ij}\frac{\partial p_j}{\partial q_j}. \tag{2.51}$$

Equating marginal revenue along each out-link determines the following proposition.

Proposition 2.3. *Let $\boldsymbol{s}^*$ be an equilibrium. For all $i \in I$,*

$$p_j + q_{ij}\frac{\partial p_j}{\partial q_j} = p_k + q_{ik}\frac{\partial p_k}{\partial q_k}, \tag{2.52}$$

for all $j, k \in N_i^-(g)$.

Given that each intermediary's price is set equal to their marginal revenue, we can also establish the following corollary.

Corollary 2.2. *Let $\boldsymbol{s}^*$ be an equilibrium. For all $i \in I \setminus I_m$,*

$$p_i = p_j + q_{ij}\frac{\partial p_j}{\partial q_j}, \tag{2.53}$$

for all $j \in N_i^-(g)$.

From Equation (2.53), i's marginal revenue can be separated into the difference between two components for each buyer: the first is the buyer's price, and the second is the product of the quantity i sells to the buyer and the rate at which that buyer's price changes with quantity, $\frac{\partial p_j}{\partial q_j}$, which is negative.

In Equation (2.53), p_j is a function of q_{ij}. Again holding $\boldsymbol{q}_{-ij}$ fixed, we set p_{j-i} to be the price j would pay for the quantity of good provided by all connected sellers excluding i. That is, $p_{j-i} = p_j(0, \boldsymbol{q}_{-ij})$. Then, we may conclude that

$$p_j = p_{j-i} + q_{ij}\frac{\partial p_j}{\partial q_j}. \tag{2.54}$$

Finally, we can substitute Equation (2.54) into Equation (2.53) to find the marginal revenue that i receives from each out-link in equilibrium as a function of i's allocation, holding the actions of all other intermediaries fixed.

$$p_i = p_{j-i} + q_{ij}\frac{\partial p_j}{\partial q_j} + q_{ij}\frac{\partial p_j}{\partial q_j} \tag{2.55}$$

$$p_i = p_{j-i} + 2q_{ij}\frac{\partial p_j}{\partial q_j}. \tag{2.56}$$

Whenever intermediary i is selling good to an intermediary j, the price paid by i in equilibrium is the difference of p_{j-1}, the price j would pay if $q_{ij} = 0$, and $-2q_{ij}\frac{\partial p_j}{\partial q_j}$, i's individual effect on j's price.

Equation (2.56) must hold in equilibrium for all intermediaries $j \in N_i^-(g)$. Rearranging Equation (2.56) reveals the quantity of good that intermediary i will allocate to each $j \in N_i^-(g)$ in equilibrium,

$$q_{ij} = \frac{p_{j-i} - p_i}{-2\frac{\partial p_j}{\partial q_j}}. \tag{2.57}$$

From Equation (2.57), we see the factors that contribute to the amount of the good that will be allocated to a new link from intermediary i to j. The numerator increases when the current price being paid by intermediary j is increases, as intermediary i will receive the a greater amount of revenue from his first marginal unit allocated along the new link. The denominator will decrease as the magnitude of $\frac{\partial p_j}{\partial q_j}$ decreases, as this will mean that j's price is less sensitive to the quantity of the good that j receives from i. Any additional quantity that i allocates to i will reduce the revenue that i is already receiving from j by a smaller amount.

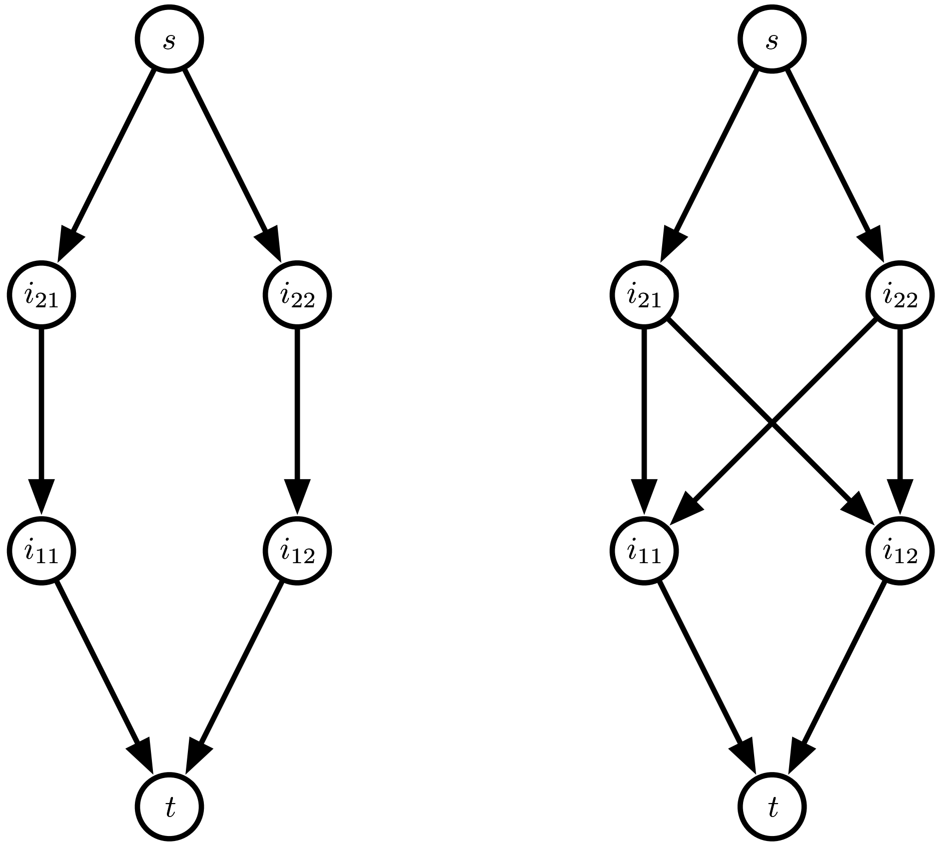
2.5 Efficiency

In this section, we evaluate which network structures are optimal for consumers, which we measure according to the consumer surplus at node t. Because we have assumed a linear inverse demand function in Equation (2.1), consumer surplus is equal to $\frac{1}{2}q_t^2$. As consumer surplus is strictly increasing in function of q_t, we may say that if the market quantities in networks g and g' are q_t and q_t', respectively, and if $q_t < q_t'$, then consumers have greater consumer surplus under network g'.

We begin by contrasting the two networks that represent a minimal and maximal level of competition between intermediaries. Define the following two networks.

Figure 2.4: Sparse and dense networks with $m = 2$ and $n = 4$

(a) Sparse network (b) Dense network



Definition 2.1 (Sparse Network). *$g^s(m, k)$ is the network with $m + 2$ tiers and k intermediaries in each tier, such that $d_i^+ = 1\, \forall\, i \in I$ and $d_i^- = 1\, \forall\, i \in I$.*

Definition 2.2 (Dense Network). *$g^c(\mathcal{P}(I))$ is the network on partition $\mathcal{P}(I)$ where, $\forall\, i \in I_j$, $d_i^+ = \left|I_{j-1}\right|$ and $d_i^- = \left|I_{j+1}\right|$.*

$g^s(m, k)$ is a network with minimal competition: every intermediary has exactly one in-link and one out-link such that there are k distinct intermediation chains of length $m + 2$ running from s to t. In contrast, $g^c(\mathcal{P}(I))$ has the maximal amount of competition: in each tier every intermediary has links to all agents in the next tier. The market quantity in these two networks is determined by the following propositions.

Proposition 2.4. *For the network $g^s(m, k)$,*

$$q_t = \frac{k}{(2^m + k - 1)}. \tag{2.58}$$

Proposition 2.5. *For the network $g^c(\mathcal{P}(I))$,*

$$q_t = \left(\prod_{k \in \{1,\ldots,m\}} \frac{|I_k|}{|I_k| + 1} \right). \tag{2.59}$$

Proposition 2.5 is consistent with Corbett and Karmarkar (Proposition 6, 2001) who solved for the optimal quantities when each tier consists of a pool of intermediaries, all of whom have equal access to the quantity of good provided by the previous tier. Their setup is equivalent to the model presented here on any dense network.

Propositions 2.4 and 2.5 demonstrate that network structure plays a key role in determining consumer surplus. For fixed m and k, the lack of competition in the sparse network ensures that intermediaries supply less of the good relative to the dense market, reducing consumer surplus. We construct a measure of the consumers' loss due to incompleteness. First, define the function $q_t(g)$ to be the amount of good received at node t under the network g.

Definition 2.3 (Price of Incompleteness)**.** *For any network $g\big(\mathcal{P}(I)\big)$ on the partition $\mathcal{P}(I)$, the* price of incompleteness *is the inverse ratio of the market quantity of the good to the market quantity of the good under a dense network. That is,*

$$\mathrm{PoI}\Big(g\big(\mathcal{P}(I)\big)\Big) = \frac{q_t\Big(g^c\big(\mathcal{P}(I)\big)\Big)}{q_t\Big(g\big(\mathcal{P}(I)\big)\Big)}. \tag{2.60}$$

As the price of incompleteness increases, consumers are less well off. PoI measures the aggregate effects of the intermediaries' market power in the network; intermediaries with market power may reduce their quantity of good sold in order to increase downstream prices, and this results in less of the good arriving at the market.

In the most extreme case, the price of incompleteness for the sparse network is equal to

$$\mathrm{PoI}\big(g^s(k,m)\big) = \frac{q_t\big(g^c(k,m)\big)}{q_t\big(g^s(k,m)\big)} \tag{2.61}$$

$$= \frac{\left(\frac{k}{k+1}\right)^m(-1+2^m+k)}{k}, \tag{2.62}$$

which follows from Propositions 2.4 and 2.5. By taking the first and second derivatives of this function we can easily show the following proposition.

Proposition 2.6. *For any fixed $k \geq 2$, for $m \geq 2$, $\mathrm{PoI}\big(g^s(k,m)\big)$ is an increasing convex function of m.*

As a result, we conclude that, for fixed-tier size, as the length of a network grows, the potential loss to consumers due to incomplete markets grows without bound at an increasing rate.

2.6 Conclusion

We have developed a model where intermediaries in a network convey a good from its source to a consumer market. Intermediaries compete in quantity, selecting the amount of the good to provide to each downstream intermediary. The price received for the good is equal to a downstream intermediary's willingness-to-pay, which is the marginal revenue that they will receive when they in turn sell the good further downstream.

In contrast to posted price models, intermediaries will not always seek the highest price available. Because price decreases as quantity increases, intermediaries may increase the revenue they receive from a downstream neighbour by lowering the quantity that neighbour receives. As such, an optimal decision for any intermediary will involve distributing their quantity of good across their available buyers. When an intermediary has exclusive access to a buyer, their ability to determine downstream prices is greatest, and this ability diminishes when they must compete with other sellers.

We have found an analytical solution to our intermediation chain model. While there is some prior literature that incorporates quantity competition into supply chains, we are only aware of models where the complexity has required that the models be solved numerically. The existence of an analytical solution allows for increased ability to analyse the interactions between players and the dynamics of the model.

We have discussed the role of competition in the model, and how competition determines the welfare of consumers in the final market. Incomplete markets reduce consumer welfare, as incompleteness provides opportunities for intermediaries to withhold the good to raise prices and increase their revenues. To measure this effect, we devised a measure called the price of incompleteness, and showed that as intermediation chains get longer the maximum potential price of incompleteness increases at an increasing rate.

Chapter 3

Games on Multi-Layer Networks

3.1 Introduction

At work, school, or in our communities, we will often have opportunity to share the work of others for our own benefit. We get to enjoy the sights and smells when our neighbours plant their gardens, and just one co-author's brilliant insight may be enough to push a group project forward. These are instances of non-excludable local public goods, where individual contributions are shared by anyone with access to them. Our research is inspired by the example of research and development between firms. Firms may invest to innovate new technologies, but technological breakthroughs may quickly be adopted by other firms, and firms may be incentivised to withhold investment in the hope that others will innovate first.

Firms may have multiple research opportunities and limited resources, and maximising profit requires allocating these resources to where they are most efficient. In our model, return on investment has two factors: declining returns when multiple connected firms research the same technology, and increased costs when a firm spreads its resources widely across many technologies. Our work answers questions as to how firms, or any agents in networks, will best allocate their resources between networks.

Because firms have research links in multiple technologies, there are multiple overlapping networks in which they connect, and actions in each network are strategically determined by the linking structure and the investments of other firms in all networks. Multiple networks may be used to model many situations; for instance, as individuals we have networks of friends and networks of colleagues, with each relationship providing a different set of costs and rewards. Nearly all prior network literature involves agents in a single network; thus, we believe research connecting strategic decision making in multiple networks is novel, and will help to open new ways of thinking about how agents connect.

There is one set of agents with two public goods to invest in. Each good has a distinct set of connections describing the pairs of agents who share benefits, and benefits for any agent depend solely upon the total investment of all agents they share links with in a good. Because of the cost of investing, an agent will always prefer a neighbour's investment to an equal amount of their own investment. Firms will, when possible, seek to avoid investment when their neighbours are willing to bear the cost of investing instead.

Agents who make the choice to invest in both goods face an increased marginal cost relative to their investment in each good. This effect is labelled *distraction*, and measures inefficiency from spreading research efforts too widely. As distraction increases, firms are more heavily penalised for investing in both goods, which will incentivise them to select one good for investment. When there is no distraction, the investment decisions for each good are independent, and the model nests the work of Bramoullé and Kranton (2007).

Because marginal benefit decreases when neighbours invest, two linked agents are strategic substitutes. As well, because distraction decreases investment payoffs, an agent's two investments are strategic substitutes. We show how each of these factors affects equilibrium, and prove the existence in any network of equilibria where any investing agent receives no investment help from her neighbours.

Welfare is measured according to two factors: aggregate payoff and the distribution of payoffs. Owing to each agent's self-interested decision making, investment will always be less than an aggregate payoff maximising level, as the externalities from investment are always positive. We discuss how an agent's neighbours affect his choice of which good to invest in, which can factor into an agent selecting the wrong investment good with respect to aggregate welfare. We establish that agents who invest in both goods will always be the least well-off agents in any equilibrium.

Increasing the cost of investment will decrease the payoff generated by any agent who invests in both layers, when holding the investments of all agents fixed. However, strategic implications when agents act in response to an increase in distraction may provide benefit to these investors. As making two investments becomes unprofitable, investors are forced to choose a single good for investment. This raises minimum payoffs in a network, as dual-investors fair worst, and, so long as there are other investors to replace the lost investment, the efficiency of investments can rise. Adding links is often beneficial for aggregate payoff, as new links spread investment benefits, but some links may reduce aggregate payoff if they connect investors who respond by lowering their investments.

For an equilibrium to be stable—when a sequence of myopic best responses by all agents to a small perturbation of equilibrium converges to the original equilibrium—requires additional constraints on the connections of any non-investing agent. As well, these constraints are more strict when a non-investing agent is only linked to investors who invest in both goods. This leads to the conclusion that the subset of equilibria that are stable will be the most equitable equilibria, as they will contain a higher proportion of investors in one good versus those who invest in none or both.

We discuss an adaption of the model that, while simplifying payoff structure, allows for a much broader set of strategic interactions. On each layer, actions between agents may be strategic complements or substitutes, and this flexibility applies also to the relationship between an agent's two actions. In this simplified extension, we show the parameter space on which a unique equilibrium exists.

This chapter contributes to an extensive literature on public good investment, where public goods are generally undersupplied by voluntary contributions. Warr (1983) and Bergstrom et al. (1986) show that aggregate contribution and individual consumption are invariant to transfers between contributing agents, provided that transfers leave the level of consumption of all contributing agents above their original level of private consumption. Network models allow for local public goods, where benefits are shared only by agents connected to contributors (e.g. Allouch 2015; Allouch 2017; Bramoullé and Kranton 2007). Elliott and Golub's (2019) model has one universal public good and a weighted network describing heterogeneous inter-agent benefits from contribution. Foster and Rosenweig (1995) empirically show that knowledge does spread through a network, but more slowly than it would if individuals were considering their neighbours. In contrast to these models, our model is novel because it has two public goods on two networks; each good is underinvested in, and an agent's choice of good may have positive or negative externalities for neighbours.

We contribute to the study of strategic decision-making in networks. In these games, the connections between agents determine the strategic effects that their actions have on each other. The following surveys: Bramoullé and Kranton (2016) and Jackson and Zenou (2015), and books: Goyal (2007) and Jackson (2008), provide a starting point for this literature. Linear quadratic payoff models are highly flexible, allowing for a wide range of strategic effects through the adjustment of a single parameter (e.g. Ballester, Calvó-Armengol et al. 2006; Ballester, Zenou et al. 2010; Calvó-Armengol and Zenou 2004). Bramoullé, Kranton and D'Amours's (2014) generic model nests many of these games, and they analyse what drives games of strategic substitutes to have corner solutions and multiple equilibria. Galeotti et al. (2010) find that when agents in games of strategic substitutes have incomplete network information and act upon their expected location in a network, unique equilibria exist. By connecting two networks in our model, we show that the outcome in one network determines the set of potential outcomes in the other.

Our model helps to explain a problem of allocating resources in a network. When agents have a fixed budget to invest, they are unable to invest in all profitable opportunities and must find the set of investments that is most efficient (e.g. Baumann 2015; Bloch and Dutta 2009; Salonen 2014). In contrast, in our model an agent's budget is unconstrained, but the choice of taking multiple investments makes each investment less profitable. Thus, a profit optimising agent may be forced to choose between two network actions that are independently profitable because they are strategic substitutes.

We contribute to the understanding of how network links may create or reinforce inequality. Dalton (1920) and Atkinson (1970) explore which measures of dispersion in a population best capture inequality. In Gagnon and Goyal's (2017) model, agents have a network action and a market action. Taking the market action changes the network payoffs; and when poorer agents in the network receive greater benefit from the network action, inequality falls. In our model, both actions are network actions, but costly actions in one network provide benefits in the other, which can reduce inequality.

Our model provides insight into the costs, benefits and strategies induced by R&D networks. In some models (e.g. Goyal, Konovalov et al. 2008; Goyal and Moraga-González 2001) cost reduction is greatest when firms cooperate, and prior to market competition research spending is complementary. In Westbrock (2010), links provide fixed R&D benefits. Our model follows most closely the assumptions of Bramoullé and Kranton (2007), that one firm's research may substitute for another firm's research, and because firms do not consider the benefits they provide to their neighbours they will invest below a level that is efficient.

This chapter contributes a new model of multi-layer networks. There are few existing papers where agents interact concurrently in multiple networks. König et al. (2014) model firms who compete in local markets after making cost-reducing investments in R&D networks. The price-determining markets are modelled as overlapping coalitions. Chen et al. (2018) focus primarily on a single layer model, but they provide an extension where agents are connected in two networks with an action on each network, where strategic network effects are complements. Joshi et al. (2019) have a model where agents begin in a fixed network. They then form links to create a second network, where the benefits from network positioning are jointly derived from the two networks. Excluding our own model, we are not aware of any multi-layer models where inter-agent actions on both layers may be strategic substitutes.

This chapter proceeds as follows: Section 3.2 presents the model. Section 3.3 provides analysis of equilibrium, welfare properties, comparative statics, and stable equilibrium. Section 3.4 includes further discussion of the model and its implications. Section 3.5 discusses potential extensions. Section 3.6 concludes. All proofs are provided in Appendix C.1.

3.2 Model

There are n agents, each existing in the set $N = \{1, \dots, n\}$. These agents have the opportunity to invest effort in two non-excludable, local public goods, good 1 and good 2. Each public good has a distinct set of links that describe pairs of agents who share the benefits of public good investment. The set of links for good p is g^p, $\forall\, p \in \{1, 2\}$, which contains a binary element g^p_{ij} for each pair of agents $i, j \in N$. If a link exists in good p between i and j then $g^p_{ij} = 1$, otherwise $g^p_{ij} = 0$. Each set of links will be referred to as a separate *layer* of the network. Each layer is undirected, meaning that a link between agents i and j in layer p is a link between j and i, and $g^p_{ij} = g^p_{ji}\, \forall\, i, j \in N$, $\forall\, p \in \{1, 2\}$. As well, we assume that an agent does not link to herself, implying $g^p_{ii} = 0\, \forall\, i \in N$, $\forall\, p \in \{1, 2\}$. In $g + g^p_{ij}$, $g^p_{ij} = 1$ and all other links are as in g. Similarly, in $g - g^p_{ij}$, $g^p_{ij} = 0$ and all other links are as in g. The matrix whose i, j^{th} element is g^p_{ij} will be denoted $\boldsymbol{G}^p$.

In the layer g^p, any agent sharing a link with agent i is *connected* to i in g^p, and the set of all agents connected to agent i are i's *neighbours* in g^p, denoted $N_i(g^p) = \{j \in N \mid g^p_{ij} = 1\}$. Agent i's *neighbourhood* in g^p is the union of i's neighbours and i. Agent i's *cardinality* in g_p is the the number of neighbours that agent i has in g^p, denoted $\eta^p_i = \left|N_i(g^p)\right|$. An agent with no neighbours in g^p is considered to be in *autarky*.

Agents choose to invest in one, both, or neither of the public goods. An agent i's investment is $s_i = (s^1_i, s^2_i) \in S$, where S is a convex subset of $\mathbb{R}^2_+$ that includes the investment $(0, 0)$. The profile of all investments in the network is the two-dimensional vector $s = (s_1, \ldots, s_n) \in S^n$.

In each layer, the network structure and action set are consistent with many existing network games. While there are a small number of models addressed in Section 3.1 that include aspects of two networks, our model is the only model we are aware of that combines two actions on two separate layers into a multi-layer network with a single payoff function.

The payoff of any agent is defined by the following function:

$$\Pi_i(s \mid g) = f\left(s^1_i + \sum_{j \in N_i(g^1)} s^1_j\right) - cs^1_i + f\left(s^2_i + \sum_{j \in N_i(g^2)} s^2_j\right) - cs^2_i - \beta s^1_i s^2_i. \tag{3.1}$$

The benefit function f is twice-differentiable and strictly concave, with $f(0) = 0$, $f'(\cdot) > 0$, $f''(\cdot) < 0$, and $f'(0) > c$. Because f is the same in both layers, comparative analysis is restricted to differences in the linking structure between the two layers. However, extending the model to allow for different benefit functions is straightforward and many of the conclusions persist. $c > 0$ is a fixed cost of investment that is constant across both layers.

The term $\beta s^1_i s^2_i$ incorporates the cost of investing in two layers simultaneously. The marginal cost of investing in one layer for any agent i increases with their investment in the other layer. In a research context, this may represent an increased cost to a firm of spreading their efforts across multiple technologies. In keeping with this example, we will refer to β as a measure of *distraction*. As β increases, the cost of spreading effort across both layers increases as well. We assume that $\beta \geq 0$.

This cost term has some convenient properties. First, when $\beta = 0$, cost is additively separable into $cs^1_i + cs^2_i$, and decisions in one layer are independent from actions in the other. Second, when $s^q_i = 0$, the cost in layer g^p, cs^p_i, is independent of β—when an agent only invests in one of the two layers, their investment decision is independent of β.

Because an agent's investment provides benefit to all of his neighbours, this is a game of positive externalities. As well, if agent i makes an investment in layer g^p, and i and j are neighbours in g^p, then j's marginal benefit from investment in g^p will fall. Thus, for neighbours in layer g^p, investments in g^p are strategic substitutes. Because an agent's decision to invest in one layer increases the marginal cost for that agent of investment in the other, an agent's two investment opportunities are strategic substitutes for one another. To distinguish between these two different strategic effects, we refer to the investments of two agents connected in a layer as *inter-agent* strategic substitutes, whereas a single agent's two investments are *intra-agent* strategic substitutes.

The degree of substitution between an agent's two investments increases with β. When $\beta = 0$, the two layers are *disjoint*, equilibrium decisions in one layer are independent of equilibrium decisions in the other layer. As $\beta \to \infty$, agents will be unable to invest in both layers, and each agent must choose at most one layer to invest in. On intermediate values of β, agents may select investment in both layers, but the additional costs from the substitution effect of β may make them less likely to do so.

3.3 Analysis

Our model gives rise to four main questions: First, does equilibrium exist, and can we characterise the behaviour of all agents in equilibrium? Second, what are the welfare properties in equilibrium, measured both by aggregate payoff and the distribution of payoffs in the population? Third, how do equilibrium and welfare change with the model's parameters, specifically distraction and the linking structure? Finally, under what conditions do stable equilibria exist, and what are the welfare properties of stable equilibria?

As we have highlighted, when $\beta = 0$, an agent's two investment choices are independent, and the problem of maximising payoff for any agent is separable into maximising payoff on each layer. Thus, by setting $\beta = 0$, our model nests a base case presented by Bramoullé and Kranton (2007). We will provide comparison of our new results to this base case, but will not repeat their results in this chapter.

3.3.1 Equilibrium

The equilibrium concept used is Nash equilibrium. A strategy profile s^* is a Nash equilibrium if, for any agent i, strategy s_i^* is a strategy that maximises i's payoff, given all other agents invest according to s^*. More formally, s^* is a Nash equilibrium if

$$\Pi_i(s_i^*, s_{-i}^* \mid g) \geq \Pi_i(s_i, s_{-i}^* \mid g),\ \forall\, s_i \in S,\ \forall\, i \in N, \tag{3.2}$$

where $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ denotes the profile of investments by all agents excluding i.

We divide agents in equilibrium into three types: an agent who invests in both layers is a *dual-actor*, an agent who invests in only one layer is a *single-actor*, and an agent who does not invest at all is a *free-rider*. As well, an investor may be classified as a *specialist* if they are providing all of the investment in their neighbourhood, or as an *intermediate investor* if they are investing along with one or more neighbours. An equilibrium is *specialised* if all investors are specialists, *distributed* if there are no specialists, and a *hybrid* if it does not conform to either preceding category.

While S may be unbounded, for analysis we need only consider the feasible action space, $\tilde{S}$, defined to be the set of all $s_i \in S$ that may be optimal for an agent i—meaning that for some network (N, g) and set of actions s_{-i}, s_i may maximise i's payoff. The concavity of f ensures that $\tilde{S}$ is a compact subset of S. On this set, we make the following assumption.

Assumption 3.1. $f''(s_i^1)f''(s_i^2) > \beta^2 \; \forall \, s_i \in \tilde{S}$

Assumption 3.1 is sufficient to guarantee that the payoff function $\Pi_i(s \mid g)$ is always concave on $\tilde{S}$. This, in turn, is used to show that any agent's optimal action on any network is uniquely determined by the actions of all other agents. We begin with the following theorem.

Theorem 3.1 (Existence)**.** *Assume that Assumption 3.1 holds. On any network* (N, g) *there exists a specialist Nash equilibrium.*

The importance of specialist equilibria is reinforced in Section 3.3.4, where we discuss stable equilibria—equilibria for which a series of myopic best responses to a sufficiently small perturbation will converge on the original equilibria. There, we will show that stable equilibria must be specialist equilibria.

The proof of Theorem 3.1 relies on a complete characterisation of how all agents must act in equilibrium, which follows in Section 3.3.1. We prove the existence of equilibrium on any network with $n = 2$, then proceed inductively. For any network (N, g), we select an arbitrary agent k, and assume the existence of a specialised equilibrium on the reduced network $(N \setminus k, g)$. First, we determine when k's best-response action to the actions of the other agents does not force any of the other agents to change their action. Next, where this is not the case, we construct a finite sequence of action changes that must terminate in a specialised equilibrium.

The feasible set: $\tilde{S}$

An agent i acting in a single-layer network will have a profitable investment opportunity if their *local investment*, the sum of all of their neighbour's investments, is less than $\hat{s}_\mathrm{s}$, where $\hat{s}_\mathrm{s}$ is the unique solution to $f'(\hat{s}_\mathrm{s}) = c$ (see Bramoullé and Kranton 2007). Agent i's investment is $s_i = \hat{s}_\mathrm{s} - \bar{s}_i$, with $\bar{s}_i = \sum_{j \in N_i(g)} s_j$. We extend this to a two-layer case by letting $\bar{s}_i^p = \sum_{j \in N_i(g^p)} s_j^p \; \forall \, p \in \{1, 2\}$, and conclude that an agent i will never invest in either layer if $\bar{s}_i^p \geq \hat{s}_\mathrm{s} \; \forall \, p \in \{1, 2\}$.

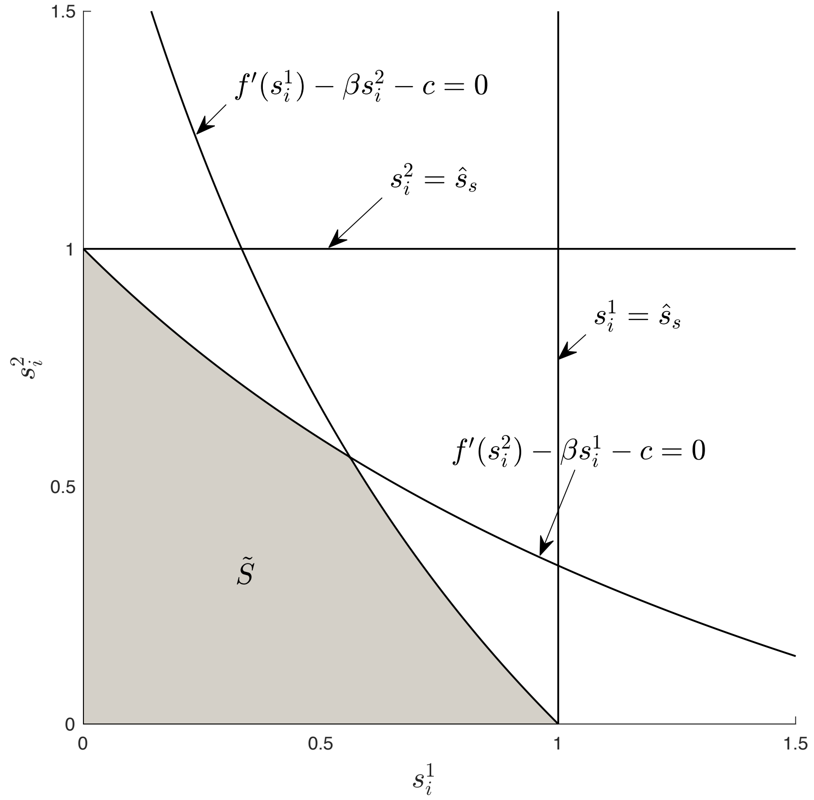
When $\bar{s}_i^p < \hat{s}_\mathrm{s}$ and $\bar{s}_i^q \geq \hat{s}_\mathrm{s}$, then i's actions are relatively straightforward. Agent i will invest only in layer g^p, making investment $s_i = \{\hat{s}_\mathrm{s} - \bar{s}_i^p, 0\}$.

When both $\bar{s}_i^p < \hat{s}_\mathrm{s}$ and $\bar{s}_i^q < \hat{s}_\mathrm{s}$, for $p \neq q$, then i's decision is more complex. The marginal benefit of an investment of s_i^p is $\frac{\partial \Pi_i(s|g)}{\partial s_i^p} = f'(s_i^p + \bar{s}_i^p) - c - \beta s_i^q$. Therefore, for any agent i in equilibrium, it must always be the case that

$$f'(s_i^1 + \bar{s}_i^1) - \beta s_i^2 - c \leq 0 \text{ and} \tag{3.3}$$

$$f'(s_i^2 + \bar{s}_i^2) - \beta s_i^1 - c \leq 0. \tag{3.4}$$

Figure 3.1: The set $\tilde{S}$, for $f(x) = 2\log(x+1)$, $c = 1$, and $\beta = \frac{1}{2}$.



Otherwise, agent i would increase investment in any layer where marginal payoff is positive. Further, we may assume that Equations (3.3) and (3.4) hold with strict equality whenever i makes positive investments in layers g^1 or g^2, respectively, and thus both equations must hold with strict equality in the case of an interior solution—when i is a dual-actor. Assumption 3.1 guarantees that the boundaries of inequalities (3.3) and (3.4) will intersect at most once. The feasible set, $\tilde{S}$, is the set constrained by inequalities (3.3) and (3.4) along with $s_i^1 \leq \hat{s}_{\mathrm{s}}$ and $s_i^2 \leq \hat{s}_{\mathrm{s}}$, as shown in Figure 3.1.

Characterisation of equilibrium

To fully describe the equilibrium actions of a single agent, we take an arbitrary agent i in a network (N, g) and let $p \in \{1, 2\}$ be the layer in which local investment for agent i is weakly lesser, and q the other layer. We have the following proposition.

Proposition 3.1. *Let i be an agent in the network (N, g), and let $\bar{s}_i^p \leq \bar{s}_i^q$ for $p, q \in \{1, 2\}$, $p \neq q$. Assume that Assumption 3.1 holds. The following four conditions must all be met in equilibrium:*

1. *If $\bar{s}_i^p \geq \hat{s}_{\mathrm{s}}$, then i will be a free-rider and make investment $s_i = (0, 0)$.*
2. *If $\bar{s}_i^p < \hat{s}_{\mathrm{s}}$ and $\hat{s}_{\mathrm{s}} - \bar{s}_i^p \geq \frac{f'(\bar{s}_i^q) - c}{\beta}$, then i will be a single-actor and make investments $s_i^p = \hat{s}_{\mathrm{s}} - \bar{s}_i^p$ and $s_i^q = 0$.*

3. *If* $\bar{s}_i^p = \bar{s}_i^q < \hat{s}_\mathrm{s}$, *then* i *will make investment* $s_i = (\tilde{s}_i, \tilde{s}_i)$, *where* $f'(\tilde{s}_i + \bar{s}_i^1) - \beta\tilde{s}_i - c = 0$.

4. *Otherwise, i's unique optimal investment must involve investment in both layers.*

Using Proposition 3.1 we may classify how any agent must act in a specialised equilibrium. Any free-rider will invest $(0,0)$, and must have local investment weakly greater than $\hat{s}_\mathrm{s}$ in each layer. A single-actor will invest $\hat{s}_\mathrm{s}$ in the layer in which no neighbours are investing, and must be connected to at least one investing agent in the other layer. Finally, if an agent is connected to no investors in either layer then they must be a dual-actor, making investment $(\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$, where $f'(\hat{s}_\mathrm{d}) - \beta\hat{s}_\mathrm{d} - c = 0$.

3.3.2 Welfare properties

The first measure of welfare we will use is the aggregate payoff of all agents, which is defined as

$$W(s \mid g) = \sum_{i \in N} \left[f\left(s_i^1 + \bar{s}_i^1\right) - cs_i^1 + f\left(s_i^2 + \bar{s}_i^2\right) - cs_i^2 - \beta s_i^1 s_i^2 \right] \tag{3.5}$$

An equilibrium profile s is *efficient* if there is no other action profile that strictly increases welfare. That is, there is no $s' \in S^n$ such that $W(s' \mid g) > W(s \mid g)$. As well, we will analyse the distribution of payoffs within a population, under the assumption that a narrower distribution is more equitable. A key measure we will use is the minimal payoff to any agent in equilibrium.

First, we consider aggregate payoffs in a network, and determine how equilibrium decisions relate to efficiency. There are two elements of an individual agent's self-interested decision making that may create a divergence from efficient outcomes. The first, where an agent underinvests relative to an efficient level in each layer, parallels the discussion of disjoint layers. The second relates to an agent's layer choice, and how this affects other agents.

Within each layer, all agents in an efficient profile who are making a positive investment must invest such that $\frac{\partial W(s|g)}{\partial s_i^p} = 0$, which implies that

$$f'\left(s_i^p + \bar{s}_i^p\right) - \beta s_i^q + \sum_{j \in N_i(g^p)} f'\left(s_j^p + \bar{s}_j^p\right) - c = 0. \tag{3.6}$$

where g^p is the layer in which i is investing and g^q is the other layer. However, in equilibrium, $f'\left(s_i^p + \bar{s}_i^p\right) - \beta s_i^q - c = 0$ for any i investing in layer g^p, and because $f'(\cdot) > 0$, the term $\sum_{j \in N_i(g^p)} f'\left(s_j^p + \bar{s}_j^p\right)$ must be strictly positive. This guarantees that any agent who invests in equilibrium will always underinvest relative to an efficient level.

In any layer of a specialised equilibrium, only non-investors may have links to investing agents, meaning the payoffs for single-actors and dual-actors are fixed. From each layer, a dual-actor will receive payoff $f(\hat{s}_{\mathrm{d}}) - c\hat{s}_{\mathrm{d}} - \frac{1}{2}\beta\hat{s}_{\mathrm{d}}^2$. A single-actor receives $f(\hat{s}_{\mathrm{s}}) - c\hat{s}_{\mathrm{s}}$ from the layer in which they are investing, and at least $f(\hat{s}_{\mathrm{d}})$ from the other layer. A free-rider must have local investment of at least $\hat{s}_{\mathrm{s}}$ in each layer, otherwise they would invest themselves, which ensures that the payoff that a free-rider receives from each layer is at least $f(\hat{s}_{\mathrm{s}})$. This leads to the following proposition.

Proposition 3.2. *In any specialist equilibrium on the network (N, g), all dual-actors will receive payoff less than that of any other agent.*

In Section 3.3.3 we examine the parameter values for which dual-actors may exist in equilibrium, concluding that parameterisations that exclude dual-actors will in turn prevent the most unequal equilibria from occurring.

3.3.3 Comparative statics

We compare the strategic implications and welfare effects of changing two variables, β and g. This is measured according to *second-best* equilibrium profiles; an equilibrium s^* is second-best if and only if there is no other equilibrium $s^{*\prime}$ such that $W(s^{*\prime} \mid g) > W(s^* \mid g)$.

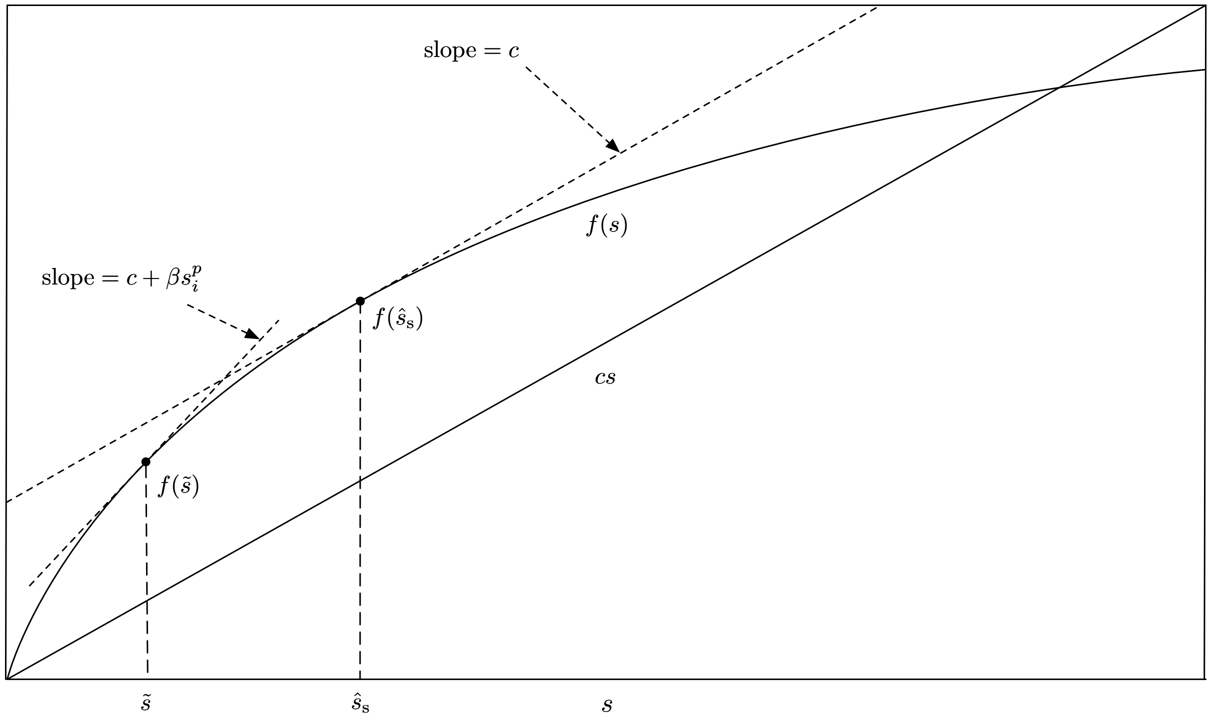
As β increases, it has multiple effects on an agent's ability to profitably invest in both layers concurrently. Directly, β affects the benefit from an agent's investments, so when β rises an agent investing in both goods will see their absolute and marginal costs increase. Holding actions constant, any agent investing in both goods will have a strictly lower payoff.

A secondary effect of an increase in β is that it expands the opportunity for agents contributing in one layer to avoid contribution in the other layer. For an agent i who makes an investment in layer g^p to not invest in g^q, his local investment in g^q, $\bar{s}_i^q$, must be sufficiently high that marginal return from any new investment will not exceed marginal costs, as is set out in Lemma C.3. When β increases, these marginal costs will increase, and the threshold level of local investment required to sustain a non-investment for i in g^q falls, expanding i's ability to free-ride in that layer.

Figure 3.2 illustrates this effect. We assume that for some agent i, $\bar{s}_i^p < \bar{s}_i^q$; thus, if i's equilibrium investment is in a single layer, i's investment will be $s_i^p = \hat{s}_{\mathrm{s}} - \bar{s}_i^p$ in layer g^p. Then the optimal local investment for i in layer g^q is $\tilde{s}$, where $f'(\tilde{s}) = c + \beta s_i^p$. For any level of local investment above $\tilde{s}$, i will not invest in layer g^q in equilibrium, whereas if local investment is below $\tilde{s}$ then i must be a dual-actor, as set out in Proposition 3.1, statement 2. As β increases, $\tilde{s}$ decreases, which increases the range of local investment in layer g^q for which i's equilibrium action is to invest in a single layer.

If s^* is a second-best equilibrium profile on the network (N, g), then the addition of the link g_{ij}^p may have three effects. If both $s_i^{p*} = 0$ and $s_j^{p*} = 0$ then $W(s^* \mid g + g_{ij}^p) = W(s^* \mid g)$. s^* is still an equilibrium profile which yields the same welfare, and there may be another equilibrium profile where either or both of $s_i^p > 0$ and $s_j^p > 0$ which yields higher welfare. If either $s_i^{p*} > 0$ or $s_j^{p*} > 0$, then

Figure 3.2: Benefit from investment



$W(s^* \mid g + g_{ij}^p) > W(s^* \mid g)$. s^* remains an equilibrium profile, and the new link passes additional benefit to a new node; thus second-best equilibrium in the new network must be strictly higher.

The final effect is where partial substitutes differ most from the case of disjoint layers. Suppose that both $s_i^{p*} > 0$ and $s_j^{p*} > 0$. Then, s^* is no longer an equilibrium in the network $(N, g + g_{ij}^p)$, and second-best welfare may increase or decrease. Holding initial investments constant, after i and j are linked in layer g^p, benefits will increase for both agents while costs will remain constant. However, because marginal benefit in layer g^p will decrease for both agents, it will no longer be an equilibrium and at least one of the agents will decrease their investment. Knock-on effects will be multiple: supposing agent j reduces s_j^p, j's marginal cost in layer g^q, $c + \beta s_j^p$, will fall, and j may also increase s_{jq} in equilibria. As well, the initial decreases in s_j^p will result in the marginal benefit increasing for all $k \in N_j(g^p)$, and these agents may then increase their investment in g^p. As these actions may effect aggregate payoff both positively or negatively, the effect of the new link g_{ij}^p is indeterminate.

The following examples shows both consequences of an additional link.

Example 3.1 (Negative Effect). *Consider the three-agent networks in Figure 3.3. We continue to use the model with $f(x) = 2\log(x+1)$, $\beta = \frac{1}{2}$, and $c = 1$. The initial network is shown in Figure 3.3a, along with each agent's investment in the second-best equilibrium, for which aggregate payoff is approximately* 5.318*. Note that in g^1, all investment is borne by the most central agent,* 1*, which would be the second-best equilibrium if the two layers were independent.*

In Figure 3.3b, a link has been added between agents 2 *and* 3 *in layer g^2. While both*

Figure 3.3: New link with a negative effect

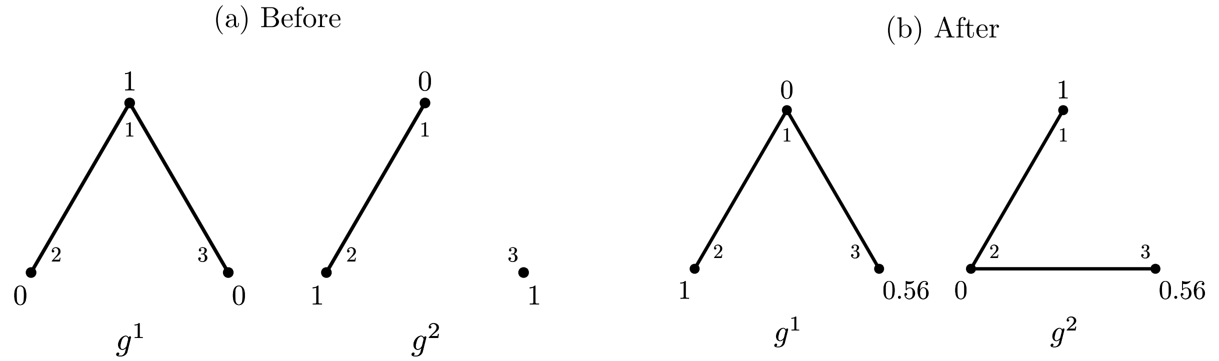
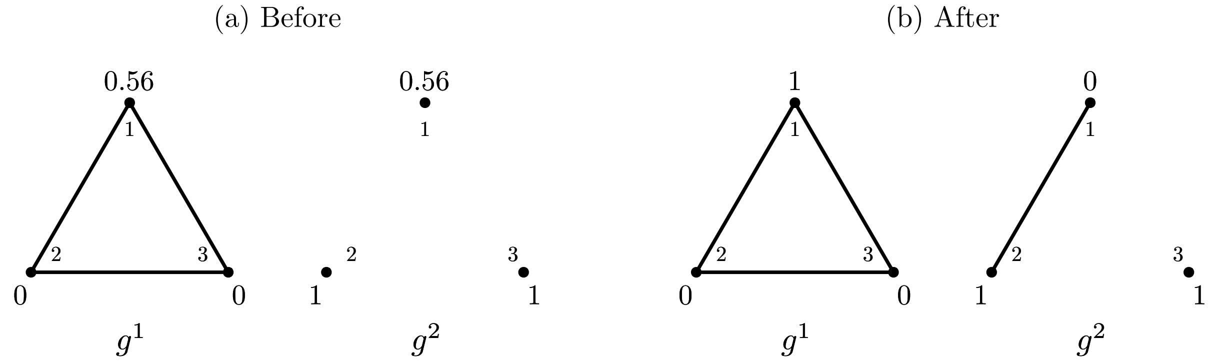


Figure 3.4: New link with a positive effect



agents had been investing before in Figure 3.3a, due to the new link they must in aggregate invest less in g^2*. This, in turn, prevents agent* 1 *from free-riding in* g^2*, and* 1 *can't bear all of the investment in* g^1*. In the second-best equilibrium, which is shown in Figure 3.3b, the new link in layer* g^2 *results in a considerably worse aggregate outcome in layer* g^1*, and the total aggregate payoff is approximately* 5.037*. Aggregate payoff in the second-best equilibrium has fallen by about* 0.2808*.*

Example 3.2 (Positive Effect)**.** *Consider the three-agent networks in Figure 3.4, and the model with* $f(x) = 2\log(x+1)$*,* $\beta = \frac{1}{2}$*, and* $c = 1$*. The initial network is shown in Figure 3.4a, along with each agent's investment in the second-best equilibrium, which has aggregate payoff of approximately* 3.057*. Because there are no links in layer* g^2*, all agents must invest in this layer in any equilibrium, and they are less able to invest in layer* g^1 *where any investment is shared.*

In Figure 3.4, a link has been added between agents 1 *and* 2 *in layer* g^2*. Then, agents* 1 *and* 2 *must reduce their aggregate investment in* g^2*, which will benefit both agents. In the second-best equilibrium shown, agent* 1 *can free-ride off of agent* 2*'s investment in* g^2*, freeing up agent* 1 *to invest a greater amount in layer* g^1 *to the benefit of all agents. Aggregate payoff is approximately* 5.318*, and the improvement in the aggregate payoff in second-best equilibrium is approximately* 2.261*.*

3.3.4 Stability of equilibrium

An equilibrium is *stable* if, after a sufficiently small perturbation of the equilibrium investment profile, a series of myopic best responses by each agent will converge back to equilibrium. Agent i's best response to the profile of all other agents' investments is defined

$$r_i(s_{-i} \mid g) = \arg\max_{s_i} \{\Pi_i(s_i, s_{-i} \mid g)\}. \tag{3.7}$$

The profile of all agents' best responses is determined by $r(s \mid g) : S^n \to S^n$. Define the series $r^t(s \mid g) = r(r^{t-1}(s \mid g) \mid g)$ with $r^0(s \mid g) = s$. Then, the equilibrium s^* is stable if there exists some $\rho > 0$ such that, for any $\boldsymbol{\epsilon} \in \mathbb{R}_+^{n\times 2}$ with $\left|\epsilon_i^p\right| < \rho$ and $s_i^{p*} + \epsilon_i^p \geq 0$, $\forall\, i \in N$, $\forall\, p \in \{1,2\}$, $\lim_{t\to\infty} r^t(s^* + \boldsymbol{\epsilon}) = s^*$.

Let s^* be an equilibrium, and suppose i is an intermediate investor in layer g^p. Then if i's neighbour in g^p increases his investment, i's best response may either be to decrease his investment in g^1 or to change his layer choice. When a permutation of equilibrium is such that investments all weakly increase in one layer, in the first step of myopic best responses all investments in the same layer will be weakly lower, while all investments in the opposite layer will be weakly higher. In every step, this pattern will reverse, and this oscillating pattern is key in demonstrating that any equilibrium with intermediate investments may be permuted in such a manner that a sequence of best responses will never converge to the original equilibrium, ensuring a stable equilibrium must be specialised. This is essential in proving the following theorem.

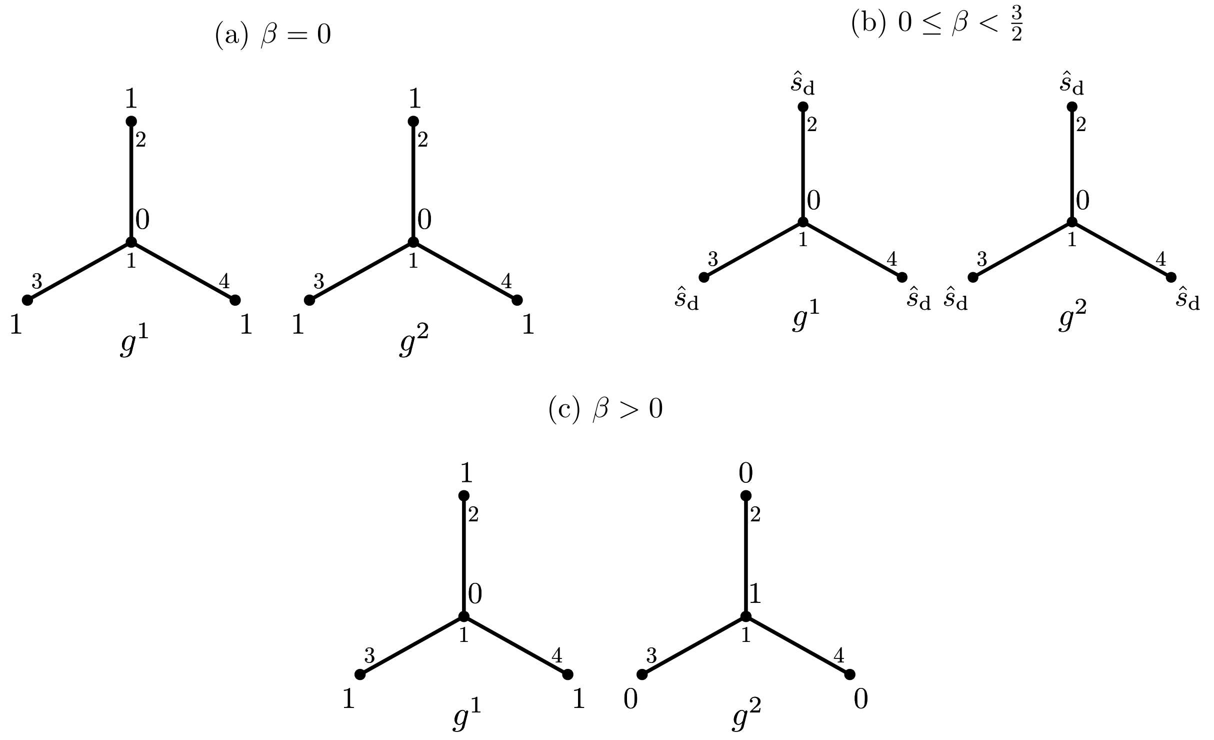
Theorem 3.2. *Assume Assumption 3.1 holds. An equilibrium is stable if and only if the set of agents N can be partitioned into four disjoint sets, L, I^1, I^2, and D, where*

1. *$\forall\, p \in \{1,2\}$, $D \cup I^p$ is a maximal independent set in layer g^p, and*
2. *$\forall\, \ell \in L$ and $\forall\, p \in \{1,2\}$, ℓ is linked in layer g^p to either*
   - *(a) more than $\frac{\hat{s}_\mathrm{s}}{\hat{s}_\mathrm{d}}$ agents in set D, or*
   - *(b) at least one agent in I^p and more than one agent in $D \cup I^p$,*

and the actions of all agents are as follows: $s_d = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})\ \forall\, d \in D$, $s_{i^1} = (\hat{s}_\mathrm{s}, 0)\ \forall\, i^1 \in I^1$, $s_{i^2} = (0, \hat{s}_\mathrm{s})\ \forall\, i^2 \in I^2$, and $s_\ell = (0,0)\ \forall\, \ell \in L$.

As the value of β increases, $\hat{s}_\mathrm{d}$ decreases because $f'(\hat{s}_\mathrm{d}) - \beta\hat{s}_\mathrm{d} - c = 0$. With $\hat{s}_\mathrm{s}$ fixed, the ratio $\frac{\hat{s}_\mathrm{s}}{\hat{s}_\mathrm{d}}$ then increases as β increases. As a result, as the value of β rises, Theorem 3.2 condition 2a indicates that, if a free-rider is not free-riding from at least one single-actor, the number of dual-actors they must be connected to will rise. Ultimately, distraction may have two effects on stable equilibria: the payoff for dual-actors and their neighbours will fall as the level of distraction rises, when actions remain constant, but a higher level of distraction may preclude equilibria that feature agents free-riding off of dual-actors, which will increase equity and may increase aggregate payoff, as the following example illustrates.

Figure 3.5: Stable Equilibria



Example 3.3. *Let $n = 4$, $f(x) = 2\log(x+1)$ and $c = 1$, so that*

$$\Pi_i(s \mid g) = 2\log\left(s_i^1 + \sum_{j \in N_i(g^1)} s_j^1 + 1\right) - s_i^1 + 2\log\left(s_i^2 + \sum_{j \in N_i(g^2)} s_j^2 + 1\right) - s_i^2 - \beta s_i^1 s_i^2, \quad (3.8)$$

and assume that the network is a star in each layer, with the same central agent in both layers. Then $\hat{s}_\mathrm{s} = 1$ and $\hat{s}_\mathrm{d} = \frac{\sqrt{1+6\beta+\beta^2}-(1+\beta)}{2\beta}$, which is decreasing in β. When $\beta = 0$, the only stable equilibrium is depicted in Figure 3.5a. In this equilibrium, the central agent receives $\Pi_1 \approx 5.54$, the peripheral agents (agents 2–4) receive $\Pi_{-1} \approx 0.77$, and aggregate payoff is $W \approx 7.86$. Figure 3.5b depicts the equivalent stable equilibrium when β is less than $\frac{3}{2}$. Now $\Pi_1 = 4\log(3\hat{s}_\mathrm{d} + 1)$ and $\Pi_{-1} = 4\log(\hat{s}_\mathrm{d} + 1) - 2\hat{s}_\mathrm{d}$. As there is no change in strategy apart from reducing investment, distraction makes all agents worse off. However, for values of β above this range, equilibrium 3.5b cannot persist, as the central agent's local investment is insufficient to support free-riding. Then, the equilibrium in Figure 3.5c, which is stable for all $\beta > 0$, is the only stable equilibrium. In this equilibrium, $\Pi_1 \approx 3.16$, $\Pi_{-1} \approx 1.77$, and $W \approx 8.48$. Because distraction forces agents into a different set of investments versus when layers are disjoint, distraction increases aggregate payoff and the distribution of payoffs is more equitable.

3.4 Discussion

In each layer, agents who contribute receive lower payoffs than those who do not, which is why dual-actors must be the least well off of all agents. Holding investments fixed, increasing distraction will penalise dual-actors further. However, we've shown that, as distraction increases, the minimal payoff for any agent in a network will eventually become higher when dual-action becomes unsustainable. In essence, being the lowliest agent becomes so unpalatable that these agents are forced to stop allowing their neighbours to profit at their expense in both layers, and, where possible, other agents will take their place and become investors.

When distraction rises, dual-actors gain a comparative strategic advantage over their free-riding neighbours. Because a dual-actor is distracted, the marginal benefit they receive from investing is lower. When agents have different marginal benefits in one layer, the game becomes similar in character to Allouch's (2015) local public good game with heterogeneous wealth. There, wealthier agents have greater marginal benefit from public investment, and in each local neighbourhood the wealthiest agents invest to the benefit of their poorer neighbours. In our model, after fixing the investments of all agents in one layer, agents will have heterogeneous payoff functions from investment in the other layer.

Gagnon and Goyal's (2017) model provides similar lessons in comparative advantages within networks. Agents have a binary market action that provides a fixed payoff to all, and a binary network action with increasing benefit as the number of neighbours taking this action rises. In the case of strategic substitutes, taking the market action reduces the rate at which network benefits increase. In this case, highly connected agents who are connected primarily to highly connected agents, those who are benefitting most from the network action, may choose not to take the market action, as it will reduce their network benefits. Less connected agents will have less to lose and will select the market action, and thus the market action serves to decrease inequality. In our model, the connected actions are both network actions, but benefitting from one action (or non-action) may still create disadvantages in the other network, leading to similar effects.

3.5 Extensions

Because strategic network effects are substitutes, actions tend to separate. As more agents invest in one layer, the marginal payoff to any agent investing in that layer weakly decreases. This, in effect, limits the amount of investment that any layer will receive. In contrast, some networks may feature complementary network effects. Then, as more agents invest in a layer, marginal payoff to agents investing in that layer will weakly increase, which may further draw agents to invest in that layer. We could see a pooling effect, as agents might coordinate and only invest in complementary layers when a sufficient number of their neighbours are doing so as well.

To allow for a more robust set of potential inter- and intra-agent interactions, we must find a simpler model. Chen et al. (2018) model socially connected criminals who engage in a second network activity, and we use their model as a starting point. Here, an agent's payoff is modelled according to the payoff function

$$\Pi_i(s \mid g) = \alpha^1 s_i^1 + \alpha^2 s_i^2 - \left[\frac{1}{2}\left(s_i^1\right)^2 + \frac{1}{2}\left(s_i^2\right)^2 + \beta s_i^1 s_i^2\right] + \delta \sum_{j \in N_i(g)} \left(s_i^1 s_j^1 + s_i^2 s_j^2\right). \quad (3.9)$$

The parameter $\beta \in (-1, 1)$ describes the nature of intra-agent strategic interactions, and the parameter $\delta > 0$ ensures that inter-agent strategic interactions are complements. We extend the model in such a manner that each action has its own layer of the network as follows:

$$\Pi_i(s \mid g) = \alpha^1 s_i^1 - \frac{1}{2}\left(s_i^1\right)^2 + \beta s_i^1 s_i^2 + \alpha^2 s_i^2 - \frac{1}{2}\left(s_i^2\right)^2 + \delta^1 \sum_{j \in N_i(g^1)} s_i^1 s_j^1 + \delta^2 \sum_{j \in N_i(g^2)} s_i^2 s_j^2. \quad (3.10)$$

Let $\Gamma(N, S, g)$ denote the game on network (N, g) with action set S^n and this payoff function. Note that $\frac{\partial^2 \Pi_i(s|g)}{\partial s_i^1 \partial s_i^2} = \beta$, and so $\beta \in (-1, 1)$ determines the nature of intra-agent strategic interactions. For $\beta > 0$ an agent's two actions are strategic complements, and $\beta < 0$ implies that an agent's two actions are strategic substitutes. Because $\frac{\partial^2 \Pi_i(s|g)}{\partial s_i^p \partial s_j^p} = \delta^p$, strategic interactions on the layer g^p are determined by the parameter $\delta^p \in (-1, 1)$, with $\delta^p > 0$ implying that actions on that layer are strategic complements and $\delta^p < 0$ describing actions which are strategic substitutes. In an extension, Chen et al. (2018) describe how their model may be extended to a multi-layer framework and provide existence results. Our use of this model goes beyond these results in allowing for strategic substitutes on networks and allowing for the action set to be bounded, for instance, enforcing that all actions must be positive.

Bramoullé, Kranton and D'Amours (2014) demonstrate how this new model can incorporate their public good model in Bramoullé and Kranton (2007), and we apply a similar extension to the main model in this paper. Taking $\beta = 0$, then in each layer $\frac{\partial \Pi_i(s|g)}{\partial s_i^p} = \alpha^p - s_i^p + \delta^p \bar{s}_i^p$. Setting marginal payoff equal to zero, excluding where investment is bound to be positive, implies that $r_i(s \mid g) = \max\left\{0, \alpha^p - \delta^p \bar{s}_i^p\right\}$. Setting $\alpha^p = \hat{s}_{\mathrm{s}}$ and $\delta^p = 1$, this is identical to the best-reply function in our main model when $\beta = 0$. Since agents in both models have identical best-reply functions, they must have identical sets of Nash equilibria.

To ensure the tractability of this new model, we first establish that $\Gamma(N, S, g)$ is a potential game (see Monderer and Shapley 1996). In a potential game, there exists a potential function ϕ such that, for all $s_i, s_i' \in S$ and $s_{-i} \in S^{n-1}$,

$$\phi(s_i, s_{-i}) - \phi(s_i', s_{-i}) = \Pi_i(s_i, s_{-i} \mid g) - \Pi_i(s_i', s_{-i} \mid g). \quad (3.11)$$

For a potential game, the set of Nash equilibria is isomorphic to the set of maxima and saddle points of the potential function.

Redefining s to be the vector $(s_{11},\dots,s_{n1},s_{12},\dots,s_{n2})^\mathsf{T}$ we propose the following.

Proposition 3.3. *The function*

$$\phi(s) = \begin{pmatrix}\alpha^1\mathbf{1}\\ \alpha^2\mathbf{1}\end{pmatrix}^\mathsf{T} s - \frac{1}{2}s^\mathsf{T}\left(\boldsymbol{I} - \begin{bmatrix}\delta^1\boldsymbol{G}^1 & \beta\boldsymbol{I}\\ \beta\boldsymbol{I} & \delta^2\boldsymbol{G}^2\end{bmatrix}\right)s \tag{3.12}$$

is a potential function for the game $\Gamma(N,S,g)$.

By determining the parameters under which the potential function is uniquely concave, we can then prove the following theorem.

Theorem 3.3. *Let*

$$\upsilon^p = \begin{cases}\delta^p\lambda_{\max}(\boldsymbol{G}^p) & \text{if } \delta^p > 0\\ \delta^p\lambda_{\min}(\boldsymbol{G}^p) & \text{if } \delta^p < 0.\end{cases} \tag{3.13}$$

If

$$(1-\upsilon^1) > 0 \text{ and} \tag{3.14}$$

$$\beta^2 < (1-\upsilon^1)(1-\upsilon^2) \tag{3.15}$$

then the game $\Gamma(N,S,g)$ *has a unique equilibrium on the action space* S.

This result is consistent with Chen et al. (2018, Theorem 6), which shows that when $\delta^1 > 0$, $\delta^2 > 0$, and $S = \mathbb{R}^2$, then $\min\{(1-\upsilon^1),(1-\upsilon^2)\} > |\beta|$ implies that there is a unique equilibrium. Our result allows for a more general parameter space, the ability to restrict the action space, and expands the threshold for which a unique equilibrium must exist.

3.6 Conclusion

Individuals and firms face choices in how to allocate their resources across existing opportunities. If our neighbour is willing to contribute her own resources towards our shared benefit, we may invest our own resources elsewhere and exploit our neighbour's generosity. Such strategic incentive to exploit neighbours may lead to outcomes where some people contribute and others free-ride. We explore these incentives in the context of innovation, where firms have opportunity to invest in researching two different technologies, and research achievements are shared with neighbouring firms.

In our model a group of agents is connected by two distinct sets of links, with each set describing pairs of agents who share benefit from investment in two different local public goods. Marginal benefit is declining in local investment, so agents have incentive to reduce investment when neighbours' investments increase, and inter-agent investments are strategic substitutes. When an agent invests in both goods, the cost of each investment increases, and intra-agent investments are strategic substitutes. We have shown how an increase in an agent's costs can be beneficial: when agent i's return on investment is greater than that of his neighbour j, because j is investing in the other technology, it may ensure that in equilibrium i will invest and j will benefit.

Our model provides a framework to analyse how investments in one good affect investments in the other, and to understand the resulting distribution of payoffs across the population. From each good, non-investing agents always receive higher payoff than investors, and payoff is decreasing in the level of investment. However, because investment in one good reduces the profitability of investment in the other, combining two networks and two public goods may have a tendency to balance payoffs between the two goods and increase equity. As the cost of investing in both goods increases, the conditions under which a single agent may invest in both goods cease to exist, and as a result the minimal achievable payoff in equilibrium increases.

To conclude, we will acknowledge some of our model's limitations and remark on potential areas for extension. Agents and public goods are heterogeneous only in linking structure. A more robust model might include heterogeneous wealth; and, if wealthier agents have a higher propensity to invest in the public good, this could overwhelm the strategic effects of investment. While assigning the same payoff function to each network ensures that the results reflect differences in linking structure, if each network had a different payoff function we might determine how different strategic affects cause agents to act. Because inter-agent actions are strategic substitutes, an agent's neighbours' investments push that agent towards investment in the other network; but, if actions are complements we might see agents with incentive to pool investments together in one of the networks.

Appendix A

Appendix to Chapter 1

A.1 Proofs

Proof of Proposition 1.1. Suppose the society is in state $\boldsymbol{\mu} = (1, 0, 0)$. Then, $\mu_w = 0$ implies that

$$\begin{aligned} u_h(\boldsymbol{\mu}) &= v\big(1 - \phi(\mu_w)\big) && \text{(A.1)} \\ &= v\big(1 - \phi(0)\big) && \text{(A.2)} \\ &= v(1 - 0) && \text{(A.3)} \\ &= v. && \text{(A.4)} \end{aligned}$$

Then by Assumption 1.1,

$$\begin{aligned} u_h(\boldsymbol{\mu}) &> \ell && \text{(A.5)} \\ &\geq \ell - t && \text{(A.6)} \\ &= u_f(\boldsymbol{\mu}). && \text{(A.7)} \end{aligned}$$

Similarly, $\mu_w = 0$ implies that

$$\begin{aligned} u_w(\boldsymbol{\mu}) &= t\frac{\mu_f}{\mu_w} && \text{(A.8)} \\ &= t\frac{0}{\mu_w} && \text{(A.9)} \\ &= 0 && \text{(A.10)} \\ &< u_h(\boldsymbol{\mu}). && \text{(A.11)} \end{aligned}$$

Because $u_h(\boldsymbol{\mu}) > u_f(\boldsymbol{\mu})$ and $u_h(\boldsymbol{\mu}) > u_w(\boldsymbol{\mu})$, state vector $\boldsymbol{\mu}$ describes and equilibrium.

Aggregate welfare is equal to $\mu_h v + \mu_f \ell - \mu_h v \phi(\mu_w)$. Because $v > \ell$, $\boldsymbol{\mu}$ must maximise aggregate payoff.

□

Proof of Proposition 1.2. Assume that $\exists\, t \in [0, \ell]$ such that

$$v\phi\left(\frac{t}{\ell}\right) - t \geq v - \ell, \tag{A.12}$$

and pick any t for which this is true. Set $\boldsymbol{\mu} = \left(0, \frac{\ell - t}{\ell}, \frac{t}{\ell}\right)$. Then

$$\begin{aligned} u_w(\boldsymbol{\mu}) &= t\frac{\mu_f}{\mu_w} &\text{(A.13)}\\ &= t\frac{\frac{\ell-t}{\ell}}{\frac{t}{\ell}} &\text{(A.14)}\\ &= \ell - t &\text{(A.15)}\\ &= u_f(\boldsymbol{\mu}). &\text{(A.16)}\end{aligned}$$

Also,

$$\begin{aligned} u_f(\boldsymbol{\mu}) - u_h(\boldsymbol{\mu}) &= \ell - t - v\big(1 - \phi(\mu_w)\big) &\text{(A.17)}\\ &= \ell - t - v\left(1 - \phi\left(\frac{t}{\ell}\right)\right) &\text{(A.18)}\\ &= \left(v\phi\left(\frac{t}{\ell}\right) - t\right) - (v - \ell) &\text{(A.19)}\\ &\geq 0, &\text{(A.20)}\end{aligned}$$

according to Equation (A.12). Because $u_f(\boldsymbol{\mu}) = u_w(\boldsymbol{\mu})$ and $u_f(\boldsymbol{\mu}) > u_h(\boldsymbol{\mu})$, state vector $\boldsymbol{\mu}$ describes an equilibrium.

Conversely, suppose there is some vector $\boldsymbol{\mu} = (0, 1 - \alpha, \alpha)$ that describes an equilibrium for some t such that

$$v\phi\left(\frac{t}{\ell}\right) - t < v - \ell. \tag{A.21}$$

Then,

$$\begin{aligned} u_f(\boldsymbol{\mu}) = u_w(\boldsymbol{\mu}) &\implies & \ell - t = t\frac{1-\alpha}{\alpha} &\qquad\text{(A.22)}\\ &\implies & \alpha = \frac{t}{\ell}. &\qquad\text{(A.23)}\end{aligned}$$

Then,

$$\begin{aligned} v\phi\left(\frac{t}{\ell}\right) - t < v - \ell &\implies & \ell - t < v\left(1 - \phi\left(\frac{t}{\ell}\right)\right) &\qquad\text{(A.24)}\\ &\implies & \mu_f(\boldsymbol{\mu}) < \mu_w(\boldsymbol{\mu}), &\qquad\text{(A.25)}\end{aligned}$$

which is a contradiction. Thus, if Equation (A.21) is true there cannot be an equilibrium. $\square$

Proof of Proposition 1.3. Recall from Equation (1.18) that $U(\boldsymbol{\mu}) = \mu_h v + \mu_f \ell - v\mu_h \phi(\mu_w)$. For $v > \ell$, the $\boldsymbol{\mu} \in \Delta^2$ that maximises $U(\boldsymbol{\mu})$ is $\boldsymbol{\mu} = (1, 0, 0)$.

□

Proof of Proposition 1.4. From Equation (1.26),

$$\mathrm{d}f_\tau = \lambda(1 - \mu_w^* - f_\tau)\,\mathrm{d}\tau. \tag{A.26}$$

In function notation, this is

$$f'(\tau) = \lambda\big(1 - \mu_w^* - f(\tau)\big)\,. \tag{A.27}$$

We also have a fixed point for the function $f(\tau)$,

$$f(0) = 0. \tag{A.28}$$

The function that satisfies Equations (A.27) and (A.28) is

$$f(\tau) = (1 - \mu_w^*)\left(1 - \mathrm{e}^{-\lambda\tau}\right). \tag{A.29}$$

Then,

$$\begin{aligned}
U_e &= \int_0^\infty u_e(\boldsymbol{\mu}_\tau)\,\mathrm{e}^{-\delta\tau}\mathrm{d}\tau && \text{(A.30)}\\
&= \int_0^\infty u_w\big(\boldsymbol{\mu}_{\mu_w^*}\big)\,\frac{f(\tau)}{1-\mu_w^*}\mathrm{e}^{-\delta\tau}\mathrm{d}\tau && \text{(A.31)}\\
&= \int_0^\infty u_w\big(\boldsymbol{\mu}_{\mu_w^*}\big)\,\frac{(1-\mu_w^*)\big(1-\mathrm{e}^{-\lambda\tau}\big)}{1-\mu_w^*}\mathrm{e}^{-\delta\tau}\mathrm{d}\tau && \text{(A.32)}\\
&= u_w\big(\boldsymbol{\mu}_{\mu_w^*}\big)\int_0^\infty \mathrm{e}^{-\delta\tau} - \mathrm{e}^{-(\lambda+\delta)\tau}\mathrm{d}\tau && \text{(A.33)}\\
&= u_w\big(\boldsymbol{\mu}_{\mu_w^*}\big)\left(\frac{1}{\delta} - \frac{1}{\lambda+\delta}\right)\Bigg|_0^\infty && \text{(A.34)}\\
&= u_w\big(\boldsymbol{\mu}_{\mu_w^*}\big)\,\frac{\lambda}{\delta(\lambda+\delta)}. && \text{(A.35)}
\end{aligned}$$

Agent e will create a warrior elite if and only if

$$U_e > v \tag{A.36}$$

$$u_w\left(\boldsymbol{\mu}_{\mu_w^*}\right)\frac{\lambda}{\delta(\lambda+\delta)} > v \tag{A.37}$$

$$\lambda > \frac{\delta^2 v}{u_w\left(\boldsymbol{\mu}_{\mu_w^*}\right) - \delta v}. \tag{A.38}$$

□

Proof of Proposition 1.5. We prove each statement separately.

1. Suppose that ϕ_g permits an agricultural equilibrium, and that $\phi_\ell(x) \geq \phi_g(x)\ \forall\, x \in [0,1]$. Suppose that $\boldsymbol{\mu}$ describes an agricultural equilibrium using technology of violence ϕ_g, where $\mu_h = 0$. Neither $u_f(\boldsymbol{\mu})$ nor $u_w(\boldsymbol{\mu})$ make use of the technology of violence, so the utility received by type-θ_f or type-θ_w agents is the same under either ϕ_ℓ or ϕ_g.

   Now, because $\phi_\ell(x) \geq \phi_g(x)\ \forall\, x \in [0,1]$,

   $$v\big(1-\phi_\ell(\mu_w)\big) \leq v\big(1-\phi_g(\mu_w)\big)\,, \tag{A.39}$$

   and, for vector $\boldsymbol{\mu}$, the utility for type-θ_h agents is lower using ϕ_ℓ than when using ϕ_g. Because we have assumed that $\boldsymbol{\mu}$ describes an agricultural equilibrium using ϕ_g, it must be the case that $u_f(\boldsymbol{\mu}) \geq u_h(\boldsymbol{\mu})$ and $u_w(\boldsymbol{\mu}) \geq u_h(\boldsymbol{\mu})$ using ϕ_g, and by Equation (A.39) these conditions will still hold using ϕ_ℓ. Then, $\boldsymbol{\mu}$ must also describe an agricultural equilibrium using ϕ_ℓ.

2. Suppose that ϕ_g does not permit an agricultural equilibrium, and that $\phi_\ell(x) \leq \phi_g(x)\ \forall\, x \in [0,1]$.

   Suppose that $\boldsymbol{\mu}$ describes an agricultural equilibrium using technology of violence ϕ_ℓ, where $\mu_h = 0$. Neither $u_f(\boldsymbol{\mu})$ nor $u_w(\boldsymbol{\mu})$ make use of the technology of violence, so the utility received by type-θ_f or type-θ_w agents is the same under either ϕ_ℓ or ϕ_g.

   Now, because $\phi_\ell(x) \leq \phi_g(x)\ \forall\, x \in [0,1]$,

   $$v\big(1-\phi_\ell(\mu_w)\big) \geq v\big(1-\phi_g(\mu_w)\big)\,, \tag{A.40}$$

   and, for vector $\boldsymbol{\mu}$, the utility for type-θ_h agents is lower using ϕ_g than when using ϕ_ℓ. Because we have assumed that $\boldsymbol{\mu}$ describes an agricultural equilibrium using ϕ_ℓ, it must be the case that $u_f(\boldsymbol{\mu}) \geq u_h(\boldsymbol{\mu})$ and $u_w(\boldsymbol{\mu}) \geq u_h(\boldsymbol{\mu})$ using ϕ_ℓ, and by Equation (A.40) these conditions will still hold using ϕ_g. But, then, $\boldsymbol{\mu}$ must also describe an agricultural equilibrium using ϕ_g, which contradicts are initial supposition. Therefore, we may conclude that ϕ_ℓ may not permit an agricultural equilibrium.

□

Appendix B

Appendix to Chapter 2

B.1 Definition of Matrices from Proposition 2.2

In this section, we define the matrices that are described in Proposition 2.2. For ease of explanation, we will describe how each matrix is constructed in reference to the network g_s from Figure 2.2 (reproduced in Figure B.1).

From Example 2.1, the inverse demand in tier 1 is

$$\boldsymbol{p}_1 = \mathbf{1}_2 - \boldsymbol{X}_1 \boldsymbol{q}_1, \tag{B.1}$$

where

$$\boldsymbol{X}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{B.2}$$

We are defining the matrices $\boldsymbol{U}_2$, $\boldsymbol{D}_2$, and $\boldsymbol{F}_2$ such that

$$\boldsymbol{p}_2 = \mathbf{1}_2 - 2 \left[\boldsymbol{U}_2 \left(\boldsymbol{D}_2 \boldsymbol{X}_1 \boldsymbol{D}_2^{\mathsf{T}} \circ \boldsymbol{F}_2 \right)^{-1} \boldsymbol{U}_2^{\mathsf{T}} \right]^{-1} \boldsymbol{q}_2. \tag{B.3}$$

First, index the links from tier 2 to tier 1. The ordering may be arbitrary, but by convention we will order the links sequentially: first by tail, then by head. For example, the vector of links is defined

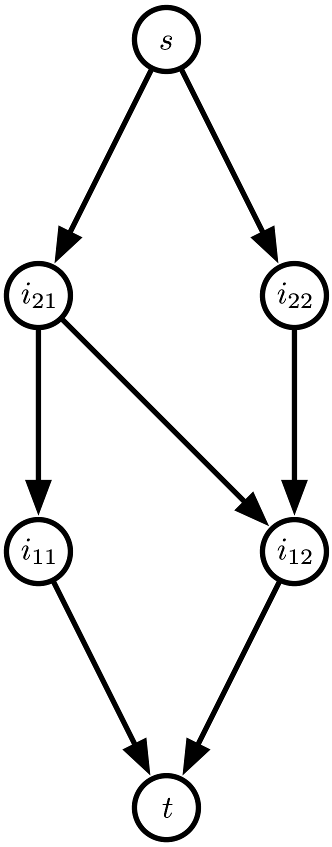
$$\boldsymbol{q}_{2,1} = \begin{pmatrix} q_{21,11} \\ q_{21,12} \\ q_{22,12} \end{pmatrix}, \tag{B.4}$$

such that $q_{21,11}$ is link 1, $q_{21,12}$ is link 2, and $q_{22,12}$ is link 3. We take the indexing of nodes in tiers 1 and 2 as their sequential ordering in their respective tiers.

The matrix $\boldsymbol{D}_2$ is defined such that

$$d_{ij} = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ link corresponds with the } j^{\text{th}} \text{ node in tier 1, and} \\ 0 & \text{otherwise.} \end{cases} \tag{B.5}$$

Figure B.1: The network g_s



For instance, in network g_s

$$\boldsymbol{D}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}. \tag{B.6}$$

For interpretation, we have defined $\boldsymbol{D}_2$ such that $\boldsymbol{p}_{2,1} = \boldsymbol{D}_2\boldsymbol{p}_1$ is the vector of prices indexed by the link ordering in $\boldsymbol{q}_{2,1}$.

In a similar manner we define $\boldsymbol{U}_2$ such that

$$u_{ij} = \begin{cases} 1 & \text{if the } j^{\text{th}} \text{ link corresponds with the } i^{\text{th}} \text{ node in tier 2, and} \\ 0 & \text{otherwise.} \end{cases} \tag{B.7}$$

For instance, in network g_s

$$\boldsymbol{U}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.8}$$

For interpretation, we have defined $\boldsymbol{U}_2$ such that, given the quantities sold along each link in $\boldsymbol{q}_{2,1}$, we may calculate the quantities at each node in tier 2 according to $\boldsymbol{q}_2 = \boldsymbol{U}_2\boldsymbol{q}_{2,1}$. The matrix $\boldsymbol{F_2}$ is defined such that

$$f_{ij} = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ link and the } j^{\text{th}} \text{ link have the same tail in tier 2, and} \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{B.9}$$

For instance, in network g_s

$$\boldsymbol{F}_2 = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}. \tag{B.10}$$

$\boldsymbol{F}_2$ relates to how intermediaries make their decisions. When two links share a tail, profits along each link flow to a single intermediary. Thus, he must take into account how his decision along one link affects his profit along another link. When intermediaries maximise profit in each stage, adjusting by the Hadamard product using $\boldsymbol{F}_2$ converts the total profit function to a potential function.

Given, $\boldsymbol{D}_2$, $\boldsymbol{U}_2$, and $\boldsymbol{F}_2$, define

$$\boldsymbol{X}_2 = 2\left[\boldsymbol{U}_2\left(\boldsymbol{D}_2\boldsymbol{X}_1\boldsymbol{D}_2^\intercal \circ \boldsymbol{F}_2\right)^{-1}\boldsymbol{U}_2^\intercal\right]^{-1} \tag{B.11}$$

$$= \begin{pmatrix} 3 & \frac{3}{2} \\ \frac{3}{2} & \frac{15}{4} \end{pmatrix}. \tag{B.12}$$

and Equation (B.3) becomes

$$\boldsymbol{p}_2 = \boldsymbol{1}_2 - \boldsymbol{X}_2\boldsymbol{q}_2. \tag{B.13}$$

The matrix $\boldsymbol{A}_2$, defined in Corollary 2.1, is

$$\boldsymbol{A}_2 = \frac{1}{2}\boldsymbol{X}_2\boldsymbol{U}_2\left(\boldsymbol{D}_2\boldsymbol{X}_1\boldsymbol{D}_2^\intercal \circ \boldsymbol{F}_2^\intercal\right)^{-1}. \tag{B.14}$$

In the network g_s,

$$\boldsymbol{A}_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{-1}{4} & 1 \end{pmatrix}. \tag{B.15}$$

$\boldsymbol{A}_2$ is the allocation matrix. Assuming that the quantities of good arriving in tier 2 are fixed and given by $\boldsymbol{q}_2$, then profit maximising intermediaries will allocate their quantities of good along each link according to $\boldsymbol{q}_{2,1} = \boldsymbol{A}_2^\intercal\boldsymbol{q}_2$.

B.1.1 Solving the Model

If there were additional tiers above tier 2, we would then solve for $\boldsymbol{X}_3$ as a function of $\boldsymbol{X}_2$, and then recursively solve until we have found $\boldsymbol{X}_x\ \forall\, x \in \{1, \ldots, m\}$. However, since tier 2 receives its good directly from the source, we instead assume that $\boldsymbol{p}_2 = \boldsymbol{0}_2$ and rearranging Equation (B.13) yields

$$\boldsymbol{q}_2 = \boldsymbol{X}_2^{-1}\boldsymbol{1}_2 \tag{B.16}$$

$$= \begin{pmatrix} \frac{1}{4} \\ \frac{1}{6} \end{pmatrix}. \tag{B.17}$$

Table B.1: Equilibrium quantities, prices, and profits for network g_s

Node	Quantity	Price	Profit
i_{11}	0.1667	0.4167	0.0278
i_{12}	0.25	0.3333	0.0625
i_{21}	0.25	0	0.0972
i_{22}	0.1667	0	0.0556
t	0.4167	0.5833	CS = 0.0868

Then,

$$\boldsymbol{q}_1 = \boldsymbol{D}_2^\mathsf{T} \boldsymbol{A}_2^\mathsf{T} \boldsymbol{q}_2 \tag{B.18}$$

$$= \begin{pmatrix} \frac{1}{6} \\ \frac{1}{4} \end{pmatrix}. \tag{B.19}$$

Equations (B.1) and (B.13) give us the pricing at each node, and using quantities and prices we may easily calculate each intermediary's profit. The solution for network g_s is presented in Table B.1.

Using the matrices defined in Appendix B.1, we obtain the same equilibrium quantities, prices, and profits as we did in Example 2.1.

B.2 Proofs

Proof of Proposition 2.1. The proof proceeds by contradiction. The price paid by any intermediary $i \in I$ is equal to their marginal revenue from q_i. Supposing $\sum_{j \in N_i^+(g)} q_{i,j} < q_i$, then i's marginal revenue from q_i is zero, and therefore $p_i = 0$. But then there could be no intermediary $k \in N_i^-(g)$ for whom $q_{k,i} > 0$, implying that $q_i = 0$. This contradicts the original supposition that $\sum_{j \in N_i^+(g)} q_{i,j} < q_i$.

□

Proof of Proposition 2.2. Consider two tiers, $x-1$ and x, with $|I_{x-1}| = \ell$ and $|I_x| = k$. Assume as well that there exists an $\ell \times \ell$ matrix $\boldsymbol{X}_{x-1}$ such that $\boldsymbol{p}_{x-1} = \boldsymbol{1}_\ell - \boldsymbol{X}_{x-1}\boldsymbol{q}_{x-1}$ is the inverse demand function in tier $x-1$. Taking matrices $\boldsymbol{U}_x$, $\boldsymbol{D}_x$, and $\boldsymbol{F}_x$ as defined in Appendix B.1. We will show that

$$\boldsymbol{p}_x = \boldsymbol{1}_k - \boldsymbol{X}_x \boldsymbol{q}_x, \tag{B.20}$$

where

$$\boldsymbol{X}_x = 2\left[\boldsymbol{U}_x \left(\boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^\mathsf{T} \circ \boldsymbol{F}_x\right)^{-1} \boldsymbol{U}_x^\mathsf{T}\right]^{-1}, \tag{B.21}$$

is the inverse demand function in tier x.

Let $r \in \{\min\{k, \ell\}, \ldots, k\ell\}$ be the number of links in $E_{x,x-1}$, and $\boldsymbol{q}_{x,x-1}$ be the vector that describes the quantity of the good that flows along each link in $E_{x,x-1}$. The matrix

$\boldsymbol{D}_x$ maps prices at each end-node in tier $x-1$ to the links in $E_{x,x-1}$. The prices along each link are described by

$$\begin{aligned}\boldsymbol{p}^{-}_{x,x-1} &= \boldsymbol{D}_x \boldsymbol{p}_{x-1} && \text{(B.22)}\\ &= \boldsymbol{D}_x \left(\boldsymbol{1}_\ell - \boldsymbol{X}_{x-1}\boldsymbol{q}_{x-1}\right) && \text{(B.23)}\\ &= \boldsymbol{1}_r - \boldsymbol{D}_x \boldsymbol{X}_{x-1}\boldsymbol{q}_{x-1}. && \text{(B.24)}\end{aligned}$$

The superscript '−' denotes the fact that these are the prices at the head of each link, whereas we would use a superscript '+' to denote the price at the tail of a link. Substituting $\boldsymbol{q}_{x-1} = \boldsymbol{D}_x^{\mathsf{T}} \boldsymbol{q}_{x,x-1}$ we get

$$\boldsymbol{p}^{-}_{x,x-1} = \boldsymbol{1}_r - \boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^{\mathsf{T}} \boldsymbol{q}_{x,x-1}. \tag{B.25}$$

Now, given an allocation $\boldsymbol{q}_{x,x-1}$, the aggregate profit for all agents in tier x is

$$\begin{aligned}\Pi_x(\boldsymbol{q}_{x,x-1}) &= \boldsymbol{q}^{\mathsf{T}}_{x,x-1}\boldsymbol{p}^{-}_{x,x-1} - C(\boldsymbol{q}_x) && \text{(B.26)}\\ &= \boldsymbol{q}^{\mathsf{T}}_{x,x-1}\boldsymbol{1}_r - \boldsymbol{q}^{\mathsf{T}}_{x,x-1}\boldsymbol{D}_x \boldsymbol{X}_{x-1}\boldsymbol{D}_x^{\mathsf{T}}\boldsymbol{q}_{x,x-1} - C(\boldsymbol{q}_x), && \text{(B.27)}\end{aligned}$$

where $C(\boldsymbol{q}_x)$ is the total amount paid for the good by agents in tier x.

We may subdivide the profit $\Pi_x(\boldsymbol{q}_{x,x-1})$ along each link according to $\Pi_x(\boldsymbol{q}_{x,x-1}) = \sum_{e\in E_{x,x-1}} \pi_e(\boldsymbol{q}_{x,x-1})$. When an intermediary $i_{xv} \in I_x$ changes the quantity $q_{xv,(x-1)w}$—the amount of good that they are providing to intermediary $(x-1)w \in I_{x-1}$—the effect on total profit is

$$\frac{\partial \Pi_x(\boldsymbol{q}_{x,x-1})}{\partial q_{xv,(x-1)w}} = \sum_{e\in E_{x,x-1}} \frac{\partial \pi_e(\boldsymbol{q}_{x,x-1})}{\partial q_{xv,(x-1)w}}. \tag{B.28}$$

The effect on intermediary i_{xv}'s individual profit is

$$\frac{\partial \pi_{xv}(\boldsymbol{q}_{x,x-1})}{\partial q_{xv,(x-1)w}} = \sum_{e\in L^{-}_{xv}} \frac{\partial \pi_e(\boldsymbol{q}_{x,x-1})}{\partial q_{xv,(x-1)w}}, \tag{B.29}$$

and the difference is

$$\frac{\partial \Pi_x(\boldsymbol{q}_{x,x-1})}{\partial q_{xv,(x-1)w}} - \frac{\partial \pi_{xv}(\boldsymbol{q}_{x,x-1})}{\partial q_{xv,(x-1)w}} = \sum_{e\in E_{x,x-1}\setminus L^{-}_{xv}} \frac{\partial \pi_e(\boldsymbol{q}_{x,x-1})}{\partial q_{xv,(x-1)w}}. \tag{B.30}$$

When i_{xv} changes the amount of good provided to an arbitrary intermediary in tier $x-1$, the difference between the change in i_{xv}'s profit and the total profit is the change in profit along all edges that do not originate at i_{xv}. We will create a potential function by adjusting the profit function in such a way that the the profit along these edges does not change when i_{xv} changes his action (Monderer and Shapley 1996, introduces the concept of a potential game). The potential function required is

$$\Phi_x(\boldsymbol{q}_x) = \boldsymbol{q}_x^\mathsf{T}\mathbf{1}_k - \left(\boldsymbol{A}_x^\mathsf{T}\boldsymbol{q}_x\right)^\mathsf{T}\left[\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}\right)\circ\boldsymbol{F}_x\right]\boldsymbol{A}_x^\mathsf{T}\boldsymbol{q}_x, \tag{B.31}$$

which makes use of the allocation matrix defined in Appendix B.1 to convert from quantities along each link to quantities at each node in I_x. We will solve for this allocation matrix later.

Our argument that this is a potential function makes use of the following lemma.

Lemma B.1. $\forall\, x \in \{1, \ldots, m\}$, *the matrix* $\boldsymbol{X}_x$ *is symmetric.*

Proof. First, note that by construction $\boldsymbol{F}_x$ is symmetric $\forall\, x \in \{1, \ldots, m\}$. Also, it is easy to show that $\boldsymbol{X}_1$ is always the symmetric matrix with 2s on the diagonal and 1s in every other position. As well, recall three facts: for any symmetric $\boldsymbol{H}$, $\boldsymbol{W} = \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}^\mathsf{T}$ is symmetric, the Hadamard product of any two symmetric matrices is symmetric, and the inverse of an invertible symmetric matrix is symmetric. Because the recursive definition of $\boldsymbol{X}_x$ with respect to $\boldsymbol{X}_{x-1}$ only makes use of these three operations and scalar multiplication, then $\boldsymbol{X}_x$ must be symmetric $\forall\, x$. □

As a result of Lemma B.1, we may conclude that the matrix $\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}$ in Equation (B.27) is symmetric. This further ensures that for any two links $e_1, e_2 \in E_{x,x-1}$

$$\frac{\partial \pi_{e_1}(\boldsymbol{q}_{x,x-1})}{\partial q_{e_2}} = \frac{\partial \pi_{e_2}(\boldsymbol{q}_{x,x-1})}{\partial q_{e_1}}. \tag{B.32}$$

If e_1 and e_2 share the same tail, then the profit along each link belongs to a single intermediary in I_x. The potential function must include the full effect of a change in q_{e_1} or q_{e_2} on the profit in each link. If they don't share the same tail, then the cross effects should be excluded. Due to the symmetry in Equation (B.32), this may be accomplished by halving each of the $e_1, e_2{}^{\text{th}}$ and $e_2, e_1{}^{\text{th}}$ entries in $\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}$. This is precisely what Hadamard multiplication by the matrix $\boldsymbol{F}_x$ accomplishes.

The final stage in the proof is to determine how the agents in I_x maximise their own payoff, given the quantities provided to them by $\boldsymbol{q}_x$. The result of this section is presented in Corollary 2.1.

Proof of Corollary 2.1. Each agent $i \in I_x$ faces the simultaneous optimisation problem:

$$\begin{aligned} \underset{q_{i,w}\ \forall\, w \in N_i^-(g)}{\text{maximize}} \quad & \pi_i(\boldsymbol{q}_j) \\ \text{subject to} \quad & \sum_{w \in N_i^-(g)} q_{i,w} \le q_i, \end{aligned} \tag{B.33}$$

which is equivalent to the global maximisation problem:

$$\begin{aligned} \underset{q_e\,\forall\, e \in E_{x,x-1}}{\text{maximize}} \quad & \Phi_i(\boldsymbol{q}_j) \\ \text{subject to} \quad & \sum_{w \in N_i^-(g)} q_{i,w} \leq q_i \quad \forall\, i \in I_x. \end{aligned} \tag{B.34}$$

We solve this problem using Karush-Kuhn-Tucker optimisation. The Lagrangian for the problem is

$$\mathcal{L}(\boldsymbol{q}_{x,x-1}, \boldsymbol{\mu}) = \Phi_i(\boldsymbol{q}_x) + \sum_{i \in I_x} \mu_i \left[\left(\sum_{w \in N_i^-(g)} q_{i,w} \right) - q_i \right] \tag{B.35}$$

If we set the first order conditions with respect to each element of the vectors $\boldsymbol{q}_{x,x-1}$ and $\boldsymbol{\mu}$ to zero, the resulting system of equations is

$$\begin{pmatrix} \mathbf{1}_k \\ -\boldsymbol{q}_{\boldsymbol{x}} \end{pmatrix} - \begin{pmatrix} \left(\boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^\mathsf{T}\right) \circ \boldsymbol{F}_x & \boldsymbol{U}_x^\mathsf{T} \\ \boldsymbol{U}_x & \mathbf{0}_{k\times k} \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_{x,x-1} \\ \boldsymbol{\mu} \end{pmatrix} = \mathbf{0}_{r+k}. \tag{B.36}$$

$$\begin{pmatrix} \left(\boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^\mathsf{T}\right) \circ \boldsymbol{F}_x & \boldsymbol{U}_x^\mathsf{T} \\ \boldsymbol{U}_x & \mathbf{0}_{k\times k} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1}_k \\ -\boldsymbol{q}_{\boldsymbol{x}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{q}_{x,x-1} \\ \boldsymbol{\mu} \end{pmatrix} \tag{B.37}$$

We are interested in solving for $\boldsymbol{q}_{x,x-1}$ as a function of $\boldsymbol{q}_x$. According to the formula for the inverse of a block matrix, the upper right corner of the inverse of the matrix in Equation (B.37) is

$$-\left[\left(\boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^\mathsf{T}\right) \circ \boldsymbol{F}_x\right]^{-1} \boldsymbol{U}^\mathsf{T} \left\{ \boldsymbol{U} \left[\left(\boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^\mathsf{T}\right) \circ \boldsymbol{F}_x\right]^{-1} \boldsymbol{U}^\mathsf{T} \right\}^{-1}. \tag{B.38}$$

Computationally, we may confirm that the sum of the row elements in the upper left block matrix of the inverse of the matrix in Equation (B.37) is always zero, so we may ignore the $\mathbf{1}_k$ in Equation (B.37). Thus, we conclude that, given $\boldsymbol{q}_x$, the agents in I_x will optimally allocate the good to each link according to

$$\boldsymbol{q}_{x,x-1} = \left[\left(\boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^\mathsf{T}\right) \circ \boldsymbol{F}_x\right]^{-1} \boldsymbol{U}^\mathsf{T} \left\{ \boldsymbol{U} \left[\left(\boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^\mathsf{T}\right) \circ \boldsymbol{F}_x\right]^{-1} \boldsymbol{U}^\mathsf{T} \right\}^{-1} \boldsymbol{q}_x. \tag{B.39}$$

The matrix in the RHS of Equation (B.39) is $\boldsymbol{A}_x^\mathsf{T}$.

□

Finally, having determined the allocation matrix $\boldsymbol{A}_x$, we will calculate the joint inverse demand of the agents in I_x. Substituting the value for $\boldsymbol{A}_x^\mathsf{T}$ into Equation (B.31) yields

$$\begin{aligned}\Phi_x(\boldsymbol{q}_x) = {} & \boldsymbol{q}_x^\mathsf{T}\mathbf{1}_k \\ & - \left(\left[\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}\right)\circ\boldsymbol{F}_x\right]^{-1}\boldsymbol{U}^\mathsf{T}\left\{\boldsymbol{U}\left[\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}\right)\circ\boldsymbol{F}_x\right]^{-1}\boldsymbol{U}^\mathsf{T}\right\}^{-1}\boldsymbol{q}_x\right)^\mathsf{T} \\ & \left[\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}\right)\circ\boldsymbol{F}_x\right]\left[\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}\right)\circ\boldsymbol{F}_x\right]^{-1} \\ & \boldsymbol{U}^\mathsf{T}\left\{\boldsymbol{U}\left[\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}\right)\circ\boldsymbol{F}_x\right]^{-1}\boldsymbol{U}^\mathsf{T}\right\}^{-1}\boldsymbol{q}_x,\end{aligned} \tag{B.40}$$

which reduces to

$$\Phi_x(\boldsymbol{q}_x) = \boldsymbol{q}_x^\mathsf{T}\mathbf{1}_k - \boldsymbol{q}_x^\mathsf{T}\left\{\boldsymbol{U}\left[\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}\right)\circ\boldsymbol{F}_x\right]^{-1}\boldsymbol{U}^\mathsf{T}\right\}^{-1}\boldsymbol{q}_x. \tag{B.41}$$

The assumption that intermediaries in I_x are willing to pay their marginal revenue for the good yields

$$\boldsymbol{p}_x = \tfrac{\mathrm{d}\Phi_x(\boldsymbol{q}_x)}{\mathrm{d}\boldsymbol{q}_x} \tag{B.42}$$

$$= \mathbf{1}_k - 2\left\{\boldsymbol{U}\left[\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T}\right)\circ\boldsymbol{F}_x\right]^{-1}\boldsymbol{U}^\mathsf{T}\right\}^{-1}\boldsymbol{q}_x. \tag{B.43}$$

□

Proof of Proposition 2.3. Suppose that

$$p_j + q_{ij}\frac{\partial p_j}{\partial q_j} \neq p_k + q_{ik}\frac{\partial p_k}{\partial q_k}, \tag{B.44}$$

and W.L.O.G. assume that

$$p_j + q_{ij}\frac{\partial p_j}{\partial q_j} > p_k + q_{ik}\frac{\partial p_k}{\partial q_k}. \tag{B.45}$$

Then, let

$$\delta = p_j + q_{ij}\frac{\partial p_j}{\partial q_j} - \left(p_k + q_{ik}\tfrac{\partial p_k}{\partial q_k}\right). \tag{B.46}$$

Now, suppose that i makes allocations $\tilde{q}_{ij} = q_{ij} + \epsilon$ and $\tilde{q}_{ik} = q_{ik} - \epsilon$ to intermediaries j and k, respectively, where

$$0 < \epsilon < \frac{\delta}{\left|\frac{\partial p_j}{\partial q_k} - \frac{\partial p_k}{\partial q_k}\right|}. \tag{B.47}$$

Then the change in i's profit will be

$$\epsilon\left(p_j + \epsilon\frac{\partial p_j}{\partial q_j}\right) + q_{ij}\epsilon\frac{\partial p_j}{\partial q_j} - \left[\epsilon\left(p_k - \epsilon\frac{\partial p_k}{\partial q_k}\right) + q_{ik}\epsilon\frac{\partial p_k}{\partial q_k}\right] \tag{B.48}$$

$$=\epsilon\left[p_j + q_{ij}\frac{\partial p_j}{\partial q_j} - \left(p_k + q_{ik}\frac{\partial p_k}{\partial q_k}\right)\right] + \epsilon^2\left(\frac{\partial p_j}{\partial q_j} - \frac{\partial p_k}{\partial q_k}\right) \tag{B.49}$$

$$=\epsilon\delta + \epsilon^2\left(\frac{\partial p_j}{\partial q_j} - \frac{\partial p_k}{\partial q_k}\right) \tag{B.50}$$

$$=\epsilon\left[\delta + \epsilon\left(\frac{\partial p_j}{\partial q_j} - \frac{\partial p_k}{\partial q_k}\right)\right]. \tag{B.51}$$

From Equation (B.47), the term in brackets must be positive, and i's change in profit from making allocations $\tilde{q}_{ij}$ and $\tilde{q}_{ik}$ must be positive, and i's initial allocation may not be an equilibrium allocation. Then we may reject Equation (B.44) and conclude that

$$p_j + q_{ij}\frac{\partial p_j}{\partial q_j} = p_k + q_{ik}\frac{\partial p_k}{\partial q_k}, \tag{B.52}$$

for all $j, k \in N_i^-(g)$.

□

Proof of Corollary 2.2. By the parameters of the model, all intermediaries pay a price equal to their marginal revenue. From Proposition 2.3, marginal revenue is equal for i along every out-link. Then, this must be i's marginal revenue, and

$$p_i = p_j + q_{ij}\frac{\partial p_j}{\partial q_j}, \tag{B.53}$$

for all $j \in N_i^-(g)$.

□

Proof of Proposition 2.4. In the network $g^s(m,k)$, $\boldsymbol{U}_x = \boldsymbol{D}_x = \boldsymbol{I}_k\ \forall\, x \in \{1,\dots,m\}$, and

$$\boldsymbol{F}_x = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & & \vdots \\ \vdots & & \ddots & \frac{1}{2} \\ \frac{1}{2} & \cdots & \frac{1}{2} & 1 \end{pmatrix}, \tag{B.54}$$

$\forall\, x \in \{1,\dots,m\}$. Repeated applications of Proposition 2.2 yield

$$\boldsymbol{X}_m = \begin{pmatrix} 2^m & 1 & \cdots & 1 \\ 1 & 2^m & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 2^m \end{pmatrix}. \tag{B.55}$$

Finally,

$$q_t = \mathbf{1}_k^\mathsf{T} \boldsymbol{X}_m^{-1} \mathbf{1}_k. \tag{B.56}$$

This is

$$\sum_{i,j} \frac{\boldsymbol{M}_{ij}}{\det(\boldsymbol{X})} = \frac{k\Big[(2^m-1)^{k-2}(2^m+k-2)\Big] - k(k-1)(2^m-1)^{k-2}}{(2^m-1)^{k-1}(2^m+k-1)} \tag{B.57}$$

$$= \frac{k}{2^m+k-1}, \tag{B.58}$$

where $\boldsymbol{M}_{ij}$ is the $(i,j)^{\text{th}}$ minor of $\boldsymbol{X}_m$.

$\square$

Proof of Proposition 2.5. First, we assert the following lemma; which, for the case of complete networks, determines the recursive definition of the matrix $\boldsymbol{X}_x$ as a function of $\boldsymbol{X}_{x-1}$ when applying Proposition 2.2.

Lemma B.2 (Complete Stages)**.** *In network g, assume that tier x has k nodes and tier $x-1$ has ℓ nodes. As well, assume that $d_i^+ = \ell \,\forall\, i \in I_x$ and $d_i^- = k \,\forall\, i \in I_{x-1}$. Then, given $\boldsymbol{X}_{x-1}$ such that*

$$\boldsymbol{p}_{x-1} = \mathbf{1}_\ell - \boldsymbol{X}_{x-1}\boldsymbol{q}_{x-1}, \tag{B.59}$$

we may define

$$\boldsymbol{X}_x = \frac{\det(\boldsymbol{X}_{x-1})}{\sum_{i,j} C_{ij}(\boldsymbol{X}_{x-1})} \boldsymbol{S}_k \tag{B.60}$$

such that

$$\boldsymbol{p}_x = \mathbf{1}_k - \boldsymbol{X}_x\boldsymbol{q}_x, \tag{B.61}$$

where $C_{ij}(\boldsymbol{X})$ is the ij^{th} cofactor of $\boldsymbol{X}$ and $\boldsymbol{S}_k$ is the $k \times k$ matrix defined

$$s_{ij} = \begin{cases} 2 & \textit{if } i = j, \textit{ and} \\ 1 & \textit{otherwise.} \end{cases} \tag{B.62}$$

Proof. Assume that tier x has k nodes and tier $x-1$ has ℓ nodes, and that $d_i^+ = \ell \,\forall\, i \in I_x$ and $d_i^- = k \,\forall\, i \in I_{x-1}$. As well, assume that $\exists\, \boldsymbol{X}_{x-1}$ such that

$$\boldsymbol{p}_{x-1} = \mathbf{1}_\ell - \boldsymbol{X}_{x-1}\boldsymbol{q}_{x-1}. \tag{B.63}$$

By Proposition 2.2,

$$\boldsymbol{X}_x = 2\left[\boldsymbol{U}_x \left(\boldsymbol{D}_x \boldsymbol{X}_{x-1} \boldsymbol{D}_x^\mathsf{T} \circ \boldsymbol{F}_x\right)^{-1} \boldsymbol{U}_x^\mathsf{T}\right]^{-1} \tag{B.64}$$

and the matrices $\boldsymbol{D}_x$, $\boldsymbol{U}_x$, and $\boldsymbol{F}_x$ are as follows:

$$\boldsymbol{D}_x = \left.\begin{pmatrix} I_\ell \\ \vdots \\ I_\ell \end{pmatrix}\right\} \times k, \tag{B.65}$$

$$\boldsymbol{U}_x = \left.\begin{pmatrix} \mathbf{1}_\ell^\mathsf{T} & \mathbf{0}_\ell^\mathsf{T} & \dots & \mathbf{0}_\ell^\mathsf{T} \\ \mathbf{0}_\ell^\mathsf{T} & \mathbf{1}_\ell^\mathsf{T} & & \vdots \\ \vdots & & \ddots & \mathbf{0}_\ell^\mathsf{T} \\ \mathbf{0}_\ell^\mathsf{T} & \dots & \mathbf{0}_\ell^\mathsf{T} & \mathbf{1}_\ell^\mathsf{T} \end{pmatrix}\right\} \times k, \text{ and} \tag{B.66}$$

$$\boldsymbol{F}_x = \left.\begin{pmatrix} \mathbf{1}_{\ell\times\ell} & \frac{1}{2}\mathbf{1}_{\ell\times\ell} & \dots & \frac{1}{2}\mathbf{1}_{\ell\times\ell} \\ \frac{1}{2}\mathbf{1}_{\ell\times\ell} & \mathbf{1}_{\ell\times\ell} & & \vdots \\ \vdots & & \ddots & \frac{1}{2}\mathbf{1}_{\ell\times\ell} \\ \frac{1}{2}\mathbf{1}_{\ell\times\ell} & \dots & \frac{1}{2}\mathbf{1}_{\ell\times\ell} & \mathbf{1}_{\ell\times\ell} \end{pmatrix}\right\} \times k. \tag{B.67}$$

Then,

$$\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T} \circ \boldsymbol{F}_x = \left.\begin{pmatrix} \boldsymbol{X}_{x-1} & \frac{1}{2}\boldsymbol{X}_{x-1} & \dots & \frac{1}{2}\boldsymbol{X}_{x-1} \\ \frac{1}{2}\boldsymbol{X}_{x-1} & \boldsymbol{X}_{x-1} & & \vdots \\ \vdots & & \ddots & \frac{1}{2}\boldsymbol{X}_{x-1} \\ \frac{1}{2}\boldsymbol{X}_{x-1} & \dots & \frac{1}{2}\boldsymbol{X}_{x-1} & \boldsymbol{X}_{x-1} \end{pmatrix}\right\} \times k. \tag{B.68}$$

Call this matrix $\bar{\boldsymbol{X}}$. Then,

$$\bar{\boldsymbol{X}}^{-1} = \det(\bar{\boldsymbol{X}})^{-1}\mathrm{adj}(\bar{\boldsymbol{X}}). \tag{B.69}$$

Next,

$$\boldsymbol{U}_x\bar{\boldsymbol{X}}^{-1}\boldsymbol{U}_x^\mathsf{T} = \boldsymbol{U}_x\left(\det(\bar{\boldsymbol{X}})^{-1}\mathrm{adj}(\bar{\boldsymbol{X}})\right)\boldsymbol{U}_x^\mathsf{T} \tag{B.70}$$

$$= \det(\bar{\boldsymbol{X}})^{-1}\boldsymbol{U}_x\mathrm{adj}(\bar{\boldsymbol{X}})\boldsymbol{U}_x^\mathsf{T}. \tag{B.71}$$

Here, we will note that

$$\boldsymbol{U}_x\mathrm{adj}(\bar{\boldsymbol{X}})\boldsymbol{U}_x^\mathsf{T} = \sum_{i,j} C_{ij}\left(\boldsymbol{X}_{x-1}\right) \left.\begin{pmatrix} (-1)^{r+c} & \frac{1}{2}(-1)^{r+c} & \dots & \frac{1}{2}(-1)^{r+c} \\ \frac{1}{2}(-1)^{r+c} & (-1)^{r+c} & & \vdots \\ \vdots & & \ddots & \frac{1}{2}(-1)^{r+c} \\ \frac{1}{2}(-1)^{r+c} & \dots & \frac{1}{2}(-1)^{r+c} & (-1)^{r+c} \end{pmatrix}\right\} \times k, \tag{B.72}$$

where r and c denote row and column indices, respectively. Denote the matrix in the above equation $\bar{\boldsymbol{S}}$. Then,

$$\left(\boldsymbol{U}_x \mathrm{adj}(\bar{\boldsymbol{X}})\boldsymbol{U}_x^\mathsf{T}\right)^{-1} = \frac{1}{\sum_{i,j} C_{ij}\left(\boldsymbol{X}_{x-1}\right)} \det(\bar{\boldsymbol{S}})^{-1}\mathrm{adj}(\bar{\boldsymbol{S}}) \tag{B.73}$$

$$= \frac{1}{\sum_{i,j} C_{ij}\left(\boldsymbol{X}_{x-1}\right)} \det(\boldsymbol{S}_k)^{-1}\boldsymbol{S}_k. \tag{B.74}$$

This leaves us with

$$2\left[\boldsymbol{U}_x\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T} \circ \boldsymbol{F}_x\right)^{-1}\boldsymbol{U}_x^\mathsf{T}\right]^{-1} = \frac{\det(\bar{\boldsymbol{X}})}{\det(\boldsymbol{S}_k)\sum_{i,j} C_{ij}\left(\boldsymbol{X}_{x-1}\right)}\boldsymbol{S}_k; \tag{B.75}$$

however, because $\bar{\boldsymbol{X}} = \boldsymbol{S}_k \otimes \boldsymbol{X}_{x-1}$,

$$\frac{\det(\bar{\boldsymbol{X}})}{\det(\boldsymbol{S}_k)} = \det(\boldsymbol{X}_x), \tag{B.76}$$

and

$$2\left[\boldsymbol{U}_x\left(\boldsymbol{D}_x\boldsymbol{X}_{x-1}\boldsymbol{D}_x^\mathsf{T} \circ \boldsymbol{F}_x\right)^{-1}\boldsymbol{U}_x^\mathsf{T}\right]^{-1} = \frac{\det(\boldsymbol{X}_{x-1})}{\sum_{i,j} C_{ij}\left(\boldsymbol{X}_{x-1}\right)}\boldsymbol{S}_k \tag{B.77}$$

□

Because every tier in a complete network is a complete tier, Lemma B.2 can be used at every tier. Conveniently,

$$\det(\boldsymbol{S}_k) = k+1, \tag{B.78}$$

and

$$\sum_{i,j} C_{ij}\left(\boldsymbol{S}_k\right) = k. \tag{B.79}$$

Then,

$$\boldsymbol{X}_m = \left(\prod_{k\in\{1,\dots,m-1\}} \frac{|I_k|+1}{|I_k|}\right)\boldsymbol{S}_{|I_m|}. \tag{B.80}$$

Finally,

$$q_t = \mathbf{1}_{|I_m|}^\mathsf{T}\boldsymbol{X}_m^{-1}\mathbf{1}_{|I_m|} \tag{B.81}$$

$$= \prod_{k\in\{1,\dots,m\}} \frac{|I_k|}{|I_k+1|}. \tag{B.82}$$

□

Proof of Proposition 2.6. The first derivative of Equation (2.62) is

$$\frac{\left(\frac{k}{k+1}\right)^m\left((k+2^m-1)\log\left(\frac{k}{k+1}\right)+2^m\log 2\right)}{k}, \tag{B.83}$$

and the second derivative is

$$\frac{\left(\frac{k}{k+1}\right)^m \left((k+2^m-1)\log\left(\frac{k}{k+1}\right)^2 + 2^{m+1}\log(2)\log\left(\frac{k}{k+1}\right) + 2^m\log(2)^2\right)}{k}. \tag{B.84}$$

Both derivatives are greater than zero for all $k \geq 2$ and $m \geq 2$.

□

Appendix C

Appendix to Chapter 3

C.1 Proofs

Proof of Theorem 3.1. Please note that this proof requires Proposition 3.1, which is proved later in this section.

This proof proceeds by induction. First we show that, when there are only 2 agents, in any network structure there is always a Nash equilibrium. For the inductive step, we select an arbitrary node k in any network and assume that there is at least one equilibrium in the network $g \setminus k$. Assigning the actions from this equilibrium to the agents in N given the network g, we determine k's best-response actions. When k's actions have no effect on k's neighbours, then this is an equilibrium. When k's actions do effect his neighbours, we show that a cascading sequence of best responses by all newly affected nodes must be finite, ultimately ending in an equilibrium.

Base Case: Assume $n = 2$.

Let $N = \{i, j\}$. Then, there are four possible network structures, and we may describe each possible set of links using the ordered pair $g = (g^1_{ij}, g^2_{ij})$. If $g = (0, 0)$, then the investments $s_i = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$ and $s_j = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$ are an equilibrium. If $g = (0, 1)$, then the investments $s_i = (\hat{s}_\mathrm{s}, 0)$ and $s_j = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$ are an equilibrium. The case where $g = (1, 0)$ is symmetric to $g = (0, 1)$, and therefore the investments $s_i = (0, \hat{s}_\mathrm{s})$ and $s_j = (\hat{s}_\mathrm{d}, \hat{s}_\mathrm{d})$ are an equilibrium. Finally, if $g = (1, 1)$, then the investments $s_i = (\hat{s}_\mathrm{s}, 0)$ and $s_j = (0, \hat{s}_\mathrm{s})$ are an equilibrium.

We have demonstrated that a specialised equilibrium exists for all possible networks with $n = 2$.

Inductive Step: Assume that for any $n \in \mathbb{Z}, n > 2$, a specialised equilibrium exists $\forall\, |N| < n$. We must show that for any network g with n agents, a specialised equilibrium exists.

Begin with the network (N, g), with $|N| = n$, and select any arbitrary agent k. Construct the reduced network, $(N \setminus k, g)$, by removing k and any links connected to k. By our inductive assumption, there is a specialised equilibrium profile on the network $(N \setminus k, g)$. Let this profile be s^*.

Assign the actions in s^* to the agents in $N \setminus k$. Then, $N \setminus k$ may be partitioned into four disjoint sets. Let D denote the set of dual-actors, with $s_d = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})\ \forall\, d \in D$. Let I^1 denote the set of single-actors investing in layer g^1, with $s_{i^1} = (\hat{s}_{\mathrm{s}}, 0) \forall\, i^1 \in I^1$. Let I^2 denote the set of single-actors investing in layer g^2, with $s_{i^2} = (0, \hat{s}_{\mathrm{s}}) \forall\, i^2 \in I^2$. Finally, let L denote the set free-riders, with $s_\ell = (0, 0)\ \forall\, \ell \in L$.

Next, we will consider all the potential sets of neighbours that k may have in (N, g), and the actions that k will take assuming all neighbours are taking actions s^*. There are four simple cases where k's actions do not require any other agents to deviate from s^*, which are described below.

Case 1: $\exists\, i^1 \in I^1$ such that $i^1 \in N_k(g^1)$ and $\exists\, i^2 \in I^2$ such that $i^2 \in N_k(g^2)$

When k is linked to at least one single-investor in each layer, k's best response is to make investment $s_k = (0, 0)$. This will not change the local investment for any agent in $N \setminus k$ in either layer, and the action set $s^* \cup s_k$ is a specialised equilibrium on the network (N, g).

Case 2: $\exists\, i \in I^1 \cup D$ such that $i \in N_k(g^1)$ and $\forall\, j \in N_k(g^2)$, $j \in L \cup I^1$

In this case, k is linked to at least one investor in layer g^1, and k is only linked to non-investors in g^2. k's best response is to make investment $s_k = (0, \hat{s}_{\mathrm{s}})$. When k does so, the local investment of all of k's neighbours in g^1 will remain unchanged, and their actions described by s^* will remain optimal when links are added to k. When k makes the investment s_k, the local investment of all of k's neighbours in g^2 will increase. Because all of k's neighbours in g^2 are non-investors in g^2 in s^*, an increase in their local investments will ensure that non-investing remains optimal. Thus, the action set $s^* \cup s_k$ is a specialised equilibrium on the network (N, g).

Case 3: $\exists\, i \in I^2 \cup D$ such that $i \in N_k(g^2)$ and $\forall\, j \in N_k(g^1)$, $j \in L \cup I^2$

This case is symmetric to Case 2 with layers g^1 and g^2 reversed, and follows accordingly.

Case 4: $\forall\, i \in N_k(g^1)$, $i \in L \cup I^2$ and $\forall\, j \in N_k(g^2)$, $j \in L \cup I^1$

Here, k is linked only with non-investors in each layer. k therefore has no local investment in either layer, and k's optimal investment is $s_k = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})$. This action will increase the local investment for all of k's neighbours in both layers. Because all of k's neighbours were non-investors in s^*, after increasing local investment non-investing will remain optimal. Therefore, the action set $s^* \cup s_k$ is a specialised equilibrium on the network (N, g).

If none of Cases 1–4 hold, then it must be true that: after k employs his best response to all of the agents in $N \setminus k$ employing strategy s^*, there is at least one agent in $N \setminus k$ who will deviate from s^* in response to k's action. This occurs when k is linked to one (or more) dual-actors in a layer, but no single actors, and k's local investment in this layer is insufficient to support non-investment. After an investment by k, these neighbouring dual-actors must respond by reducing their own investments. This scenario may concurrently occur in both layers, for instance if k is linked to one dual-actor in each layer. Without loss of generality, we will proceed by assuming that this occurs in layer g^1 and that when all agents in $N \setminus k$ employ strategy s^*, $\bar{s}_k^1 \leq \bar{s}_k^2$.

We assume that k makes the investment $s_k = (\hat{s}_{\mathrm{s}}, 0)$. Then, alter the strategies employed by all agents according to the following sequence:

1. Assign the strategy $s^* \cup s_k$ to all agents in N.
2. The only set of agents, excluding k, for whom their currently strategy is not optimal is the set of dual-actors who are linked to k in layer g^1. Assume that all of these agents change their action to $(0, \hat{s}_{\mathrm{s}})$, so that they are now single-actors in layer g^2. This will change the subsets of N in the following way, where the subscript 'old' denotes the subsets prior to this step: $I^2 = I^2_{\mathrm{old}} \cup (D_{\mathrm{old}} \cap N_k(g^1))$ and $D = D_{\mathrm{old}} \setminus (D_{\mathrm{old}} \cap N_k(g^1))$.
3. (a) There may be free-riders who are unconnected to any single-actors in g^1 and who now have insufficient local investment to support free-riding in g^1 (because at least one dual-actor to whom they are linked in g^1 became a single-actor in layer g^2 in step 2). If any such free-riders exist, take the set L' to be a maximal independent set of these free-riders in g^1. Then, assign the action $s_{\ell'} = (\hat{s}_{\mathrm{s}}, 0)$ to all agents in L', which results in the following two new sets: $I^1 = I^1_{\mathrm{old}} \cup L'$ and $L = L_{\mathrm{old}} \setminus L'$.

   (b) If L' is non-empty in step 3a, meaning there are agents switching from the set L to the set I^1, then this may cause further agents to need to change their action. Such agents could only be dual-actors who were linked to agents in L' in layer g^1, who now find themselves linked to single-actors. If any such-dual actors exist, let them compose the set D', and assign them the action $s_{d'} = (0, \hat{s}_{\mathrm{s}})$, so that they are now members of I^2. The change in sets is $I^2 = I^2_{\mathrm{old}} \cup D'$ and $D = D_{\mathrm{old}} \setminus D'$.

   (c) If a non-empty set of dual-actors become single-actors in g^2 in step 3b, then this may force additional free-riders who are connected to these agents in layer g^1 to begin investing. This mirrors the change that occurs in step 3a. If we continue to repeat steps 3a and 3b, then we will alternate between moving agents from L to I^1 and moving agents from D to I^2. Because L is finite, this sequence of repeated action changes must eventually terminate.

4. After step 2 and repeated applications of steps 3a and 3b, the change may be

summarised: a subset of L have switched to I^1 and a subset of D have switched to I^2. The new members of I^1 will have no further effects on other members of the network: they cannot have links to other agents in I^1 or else they would not have switched in the first place. When agents move from D to I^2, however, they may affect additional agents. This would only be previous members of I^2 who are now connected only to non-investors in g^1. If there are no such agents, the current action profile is a specialised equilbrium. Otherwise, let $I^{2'}$ be a maximal independent set of these agents in g^1, and assign all members of $I^{2'}$ the new action $s_{i^{2'}} = (\hat{s}_\text{d}, \hat{s}_\text{d})$. We now have the new sets $D = D_\text{old} \cup I^{2'}$ and $I^2 = I^2_\text{old} \setminus I^{2'}$.

5. (a) The change in step 4 will only affect one new set of agents: some free-riders may no longer have sufficient local investment in g^2. If there are no such free-riders the proof is complete; otherwise let L' be a maximal independent subset of these free-riders in g^2. Change the action for all of the agents in L to make them single-actors in g^2. The new subsets are $I^2 = I^2_\text{old} \cup L'$ and $L = L_\text{old} \setminus L'$.
   (b) As in step 3b, step 5a may require dual-actors to switch to single-action in layer g^1.
   (c) As in step 3c, steps 5a and 5b must terminate after a finite sequence of repetitions. At this point, a subset of agents in L will have switched to I^2, and a subset of agents in D will have switched to I^1.
6. After steps 5a–5c, we have a symmetric scenario to that preceding step 4. If there are any members of I^1 who are no longer linked to any investors in g^2, we move them to D by changing their action to dual-action. If there are no such agents, the current action profile must be a specialised Nash equilibrium.

7. After step 6, the scenario is symmetric to that preceding steps 5a–5c. We have either reached a symmetric Nash equilibrium, or a subset of agents will have to be moved from set L to set I^1 and a subset of agents may have to be moved from set D to set I^2.
8. At this point, step 4 repeats. Either we have a specialised equilibrium, or there are members of set I^2 who must be moved to set D.

If a repeated loop of steps 4–7 ever terminates, it must do so in a specialised Nash equilibrium. Now, note that steps 5a and 7 require that agents be moved from set L into another set; otherwise the algorithm will terminate. Because agents are never moved into set L, and set L must be finite to begin with, this algorithm must eventually terminate.

Therefore, a specialised equilibrium must exist with $n + 1$ agents, and the inductive step is complete.

□

Proof of Proposition 3.1. Proposition 3.1 summarises a series of lemmas that govern how an individual agent in an equilibrium must act. The first ensures that any agent has a unique optimal investment, given the investment decisions of all other agents.

Lemma C.1. *Given the investment decisions of all other agents, s_{-i}, an agent i has a unique profit maximising investment s_i.*

Proof. The Hessian matrix of $\Pi_i(s \mid g)$ at s_i is

$$\boldsymbol{H} = \begin{bmatrix} f''(s_i^1) & -\beta \\ -\beta & f''(s_i^2) \end{bmatrix} \tag{C.1}$$

By assumption, $f''(x) < 0 \,\forall\, x \in \mathbb{R}_+$, so the first leading principal minor of $\boldsymbol{H}$ is always negative. The second leading principal minor of $\boldsymbol{H}$ is $f''(s_i^1)f''(s_i^2) - \beta^2$. Assumption 3.1 tells us that $f''(s_i^1)f''(s_i^2) > \beta^2 \,\forall\, s_i \in \tilde{S}$, which implies that the second leading principal minor is positive $\forall\, s_i \in \tilde{S}$. Because all odd leading principal minors are always negative on $\tilde{S}$, and all even leading principal minors are always positive on $\tilde{S}$, then $\boldsymbol{H}$ is negative definite $\forall\, s_i \in \tilde{S}$, which in turn implies that $\Pi_i(s \mid g)$ is concave on $\tilde{S}$. By construction, $\tilde{S}$ is compact. A concave function on a compact set must have a unique maximum.

□

Next we consider an agent's actions when it is optimal for an agent to invest in a single layer.

Lemma C.2. *In Nash equilibrium, if an agent is a single-actor, then they will always invest in a layer g^p where*

$$p = \arg\min_x \bar{s}_{ix}. \tag{C.2}$$

Proof. Assume that agent i must be a single-actor, and that i is choosing between investment in layer g^1 or layer g^2. Assume that $\bar{s}_i^p \leq \bar{s}_i^q$. As well, since investment will not be profitable in g^q when $\bar{s}_i^q \geq \hat{s}_\mathrm{s}$, assume also that $\bar{s}_i^q < \hat{s}_\mathrm{s}$.

The return that i generates in either layer g^m in excess of making no investment is

$$f(\hat{s}_\mathrm{s}) - f(\bar{s}_i^m) - c(\hat{s}_\mathrm{s} - \bar{s}_i^m). \tag{C.3}$$

Then, the difference between an investment in g^p and an investment in g^q is

$$f(\hat{s}_\mathrm{s}) - f\left(\bar{s}_i^p\right) - c\left(\hat{s}_\mathrm{s} - \bar{s}_i^p\right) - \left[f(\hat{s}_\mathrm{s}) - f\left(\bar{s}_i^q\right) - c\left(\hat{s}_\mathrm{s} - \bar{s}_i^q\right)\right] \tag{C.4}$$

$$= f\left(\bar{s}_i^q\right) - f\left(\bar{s}_i^p\right) - c\left(\bar{s}_i^q - \bar{s}_i^p\right) \tag{C.5}$$

Because $f''(\cdot) < 0$ and $f'(\hat{s}_\mathrm{s}) = c$, $f'(x) > c \,\forall\, x \in [0, \hat{s}_\mathrm{s}]$. Then, given that $\bar{s}_i^p \leq \bar{s}_i^q < \hat{s}_\mathrm{s}$, it must also follow that $f\left(\bar{s}_i^q\right) - f\left(\bar{s}_i^p\right) > c\left(\bar{s}_i^q - \bar{s}_i^p\right)$ when $\bar{s}_i^p \neq \bar{s}_i^q$, which would establish that difference between investing in g^p and g^q must be positive. A utility maximising single-actor will, therefore, always invest in a layer with minimal local effort.

□

We may determine precisely when investment in a single layer will be optimal for any agent, and also what investment must be in this scenario.

Lemma C.3. *Let s^* be an equilibrium in the network (N, g). Assume that there exists an agent i for whom $\bar{s}_i^p < \bar{s}_i^q$ and $\bar{s}_i^p < \hat{s}_{\mathrm{s}}$. Then i is a single-actor making investment $s_i^{p*} = \hat{s}_{\mathrm{s}} - s_i^p$ in layer g^p if and only if*

$$s_i^{p*} \geq \frac{f'(\bar{s}_i^q) - c}{\beta}. \tag{C.6}$$

Proof. Let $\bar{s}_i^p \leq \bar{s}_i^q$, and assume that agent i is investing in only one layer. By Lemma C.2, we know that this must be layer g^p. Then i's investment is $s_i = \{s_i^p, 0\}$.

Suppose i is making investment $s_i^p \neq \hat{s}_{\mathrm{s}} - \bar{s}_i^p$ in layer g^p. Then i's marginal payoff in layer g^p, is

$$\begin{aligned} f'(s_i^p + \bar{s}_i^p) - c &\neq f'(\hat{s}_{\mathrm{s}} - \bar{s}_i^p + \bar{s}_i^p) - c && \text{(C.7)}\\ &= f'(\hat{s}_{\mathrm{s}}) - c && \text{(C.8)}\\ &= 0 && \text{(C.9)} \end{aligned}$$

Because i's marginal payoff in layer g^p is not zero, i may improve his payoff by changing his investment, and i cannot be making an equilibrium investment. Thus, we may conclude that $s_i^p = \hat{s}_{\mathrm{s}} - \bar{s}_i^p$.

Now, given $s_i^p = \hat{s}_{\mathrm{s}} - \bar{s}_i^p$ and $s_i^q = 0$, i's marginal payoff from investment in layer g^q is $f'(\bar{s}_i^q) - \beta(\hat{s}_{\mathrm{s}} - \bar{s}_i^p) - c$. i may only invest zero in layer g^q if i's marginal payoff is weakly negative, that is

$$\begin{aligned} f'(\bar{s}_i^q) - \beta(\hat{s}_{\mathrm{s}} - \bar{s}_i^p) - c &\leq 0 && \text{(C.10)}\\ \hat{s}_{\mathrm{s}} - \bar{s}_i^p &\geq \frac{f'(\bar{s}_i^q) - c}{\beta} && \text{(C.11)} \end{aligned}$$

□

In the case where local investment for an agent is equal in both layers, then any investing agent must invest equally in both layers.

Lemma C.4. *Let s^* be an equilibrium in the network (N, g). For any agent i whose local investment is equal in both layers and less than $\hat{s}_{\mathrm{s}}$, that is $\bar{s}_i^1 = \bar{s}_i^2 < \hat{s}_{\mathrm{s}}$, i must be making investment $s_i = (\tilde{s}_i, \tilde{s}_i)$, where $f'(\tilde{s}_i + \bar{s}_i^1) - \beta\tilde{s}_i - c = 0$.*

Proof. Because $\bar{s}_i^1 = \bar{s}_i^2$, Equations (3.3) and (3.4) are equivalent, and a solution to one is a solution to both. Recall Equation (3.3):

$$f'(s_i^1 + \bar{s}_i^1) - \beta s_i^1 - c \leq 0 \tag{C.12}$$

Because $\bar{s}_i^1 < \hat{s}_{\mathrm{s}}$,

$$\begin{aligned} f'(0 + \bar{s}_i^1) - \beta 0 - c &= f'\left(\bar{s}_i^1 - c\right) && \text{(C.13)} \\ &> f'(\hat{s}_{\mathrm{s}}) - c && \text{(C.14)} \\ &> 0, && \text{(C.15)} \end{aligned}$$

and because the left side of Equation (C.12) is continuous and decreasing in $\tilde{s}_{i1}$, there must be some $\tilde{s}_i > 0$ for which $f'(\tilde{s}_i + \bar{s}_i^1) - \beta\tilde{s}_i - c = 0$.

By the construction of $\tilde{S}$, any solution to Equation (C.12) may not exceed the upper boundary of $\tilde{S}$. Then $s_i = (\tilde{s}_i, \tilde{s}_i) \in \tilde{S}$ is a solution to Equations (3.3) and (3.4). Lemma C.1 ensures that this is the unique solution to agent i's maximisation problem.

□

The following corollary is a direct result of Lemma C.4.

Corollary C.1. *Let $s^* \in S^n$ be an equilibrium of the network (N, g). If there exists an agent i who has no local investment in either layer, then i must be a dual-actor making investment $s_i^* = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})$, where $\hat{s}_{\mathrm{d}}$ is the solution to the equation $f'(\hat{s}_{\mathrm{d}}) - \beta\hat{s}_{\mathrm{d}} - c = 0$.*

Proof. From Lemma C.4, we may state that, $\forall\, x \in [0, \hat{s}_{\mathrm{s}})$, Equations (3.3) and (3.4) will have a unique symmetric solution: $s_i^* = (\tilde{s}_i, \tilde{s}_i)$, if $\bar{s}_i^1 = \bar{s}_i^2 = x$. Setting $\bar{s}_i^1 = \bar{s}_i^2 = 0$, we find that $s_i^* = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})$, where $\hat{s}_{\mathrm{d}}$ is the solution to the equation $f'(\hat{s}_{\mathrm{d}}) - \beta\hat{s}_{\mathrm{d}} - c = 0$.

□

Lemmas C.1 to C.4 and Corollary C.1 provide sufficient support for all of the claims in Proposition 3.1.

□

Proof of Proposition 3.2. The payoff of a dual-actor in a specialist equilibrium is $\Pi_d(s^* \mid g) = 2f(\hat{s}_{\mathrm{d}}) - 2c\hat{s}_{\mathrm{d}} - \beta\left(\hat{s}_{\mathrm{d}}\right)^2$. As well, since a free-rider receives payoff of at least $f(\hat{s}_{\mathrm{s}})$ from each layer, the payoff to a free-rider must satisfy $\Pi_\ell(s^* \mid g) \geq 2f(\hat{s}_{\mathrm{s}})$. Then, $\Pi_\ell(s^* \mid g) \geq 2f(\hat{s}_{\mathrm{s}}) > 2f(\hat{s}_{\mathrm{d}}) - 2c\hat{s}_{\mathrm{d}} - \beta\left(\hat{s}_{\mathrm{d}}\right)^2 = \Pi_d(s^* \mid g)$, and free-riders are always better off than dual-actors in specialised equilibria.

The payoff that a single-actor receives in the layer in which they are investing is $f(\hat{s}_{\mathrm{s}}) - c\hat{s}_{\mathrm{s}}$. In the other layer, a single-actor must be connected to at least one investing node, otherwise the single-actor would switch to dual-action. Thus, the payoff to a single-actor must satisfy

$$\Pi_i(s^* \mid g) \geq \left[f(\hat{s}_{\mathrm{s}}) - c\hat{s}_{\mathrm{s}}\right] + f(\hat{s}_{\mathrm{d}}) \tag{C.16}$$
$$> f(\hat{s}_{\mathrm{d}}) - c\hat{s}_{\mathrm{d}} + f(\hat{s}_{\mathrm{d}}) \tag{C.17}$$
$$> 2f(\hat{s}_{\mathrm{d}}) - 2c\hat{s}_{\mathrm{d}} - \beta\left(\hat{s}_{\mathrm{d}}\right)^2 \tag{C.18}$$
$$= \Pi_d(s^* \mid g) \tag{C.19}$$

and the payoff of a single-actor must be greater than the payoff of a dual-actor. Step C.17 is based on the fact that $\hat{s}_{\mathrm{s}}$ is the optimal investment amount for a single-actor, so $f(\hat{s}_{\mathrm{s}}) - c\hat{s}_{\mathrm{s}} > f(\hat{s}_{\mathrm{d}}) - c\hat{s}_{\mathrm{d}}$, with a strict inequality because f is strictly concave and thus $\hat{s}_{\mathrm{s}}$ is unique.

Thus, all dual-actors will receive payoff less than that of a single-actor or a free-rider.

□

Proof of Theorem 3.2. First, we will prove that if N can be partitioned into the four disjoint sets, L, I^1, I^2, and D, as defined in Theorem 3.2, and that the agents take the action profile s such that $s_\ell = (0,0)\ \forall\, \ell \in L$, $s_{i^1} = (\hat{s}_{\mathrm{s}}, 0)\ \forall\, i^1 \in I^1$, $s_{i^2} = (0, \hat{s}_{\mathrm{s}})\ \forall\, i^2 \in I^2$, and $s_d = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})\ \forall\, d \in D$, then this is a stable equilibrium.

Let k be the maximum degree of any agent in either g^1 or g^2.

Note that, because $f'(\hat{s}_{\mathrm{d}}) - \beta\hat{s}_{\mathrm{d}} - c = 0$, we know that $f'(\hat{s}_{\mathrm{d}}) - \beta\hat{s}_{\mathrm{s}} - c < 0$. As well, $f'(0) - \beta\hat{s}_{\mathrm{s}} - c > 0$; otherwise single-action could be a local maximum in autarky, which would violate Lemma C.1. Because $f'(\hat{s}_{\mathrm{d}} - x) - \beta(\hat{s}_{\mathrm{s}} - x) - c$ is continuous and increasing in x, there must then be some $x \in (0, \hat{s}_{\mathrm{d}})$ such that $f'(\hat{s}_{\mathrm{d}} - x) - \beta(\hat{s}_{\mathrm{s}} - x) - c = 0$.

Following similar logic, because $f'(\hat{s}_{\mathrm{d}}) - \beta\hat{s}_{\mathrm{s}} - c < 0$ and $f'(0) - \beta\hat{s}_{\mathrm{s}} - c > 0$, there is some $y \in (0, \hat{s}_{\mathrm{d}})$ such that $f'(y) - \beta\hat{s}_{\mathrm{s}} - c = 0$. And finally, let $z = \hat{s}_{\mathrm{d}} - y$.

Let $\delta = \left\lfloor \frac{\hat{s}_{\mathrm{s}}}{\hat{s}_{\mathrm{d}}} + 1 \right\rfloor$, so δ is the minimum number of dual-actors that a free-rider who is not connected to any single-actor must be connected to, according to 2a. Assume that

$$\rho < \min\left\{\frac{\hat{s}_{\mathrm{d}}}{k}, \frac{\delta\hat{s}_{\mathrm{d}} - \hat{s}_{\mathrm{s}}}{\delta k}, \frac{x}{k}, \frac{y}{k}, \frac{z}{k}\right\} \tag{C.20}$$

. Let $\boldsymbol{\epsilon}$ be any $n \times 2$ vector such that $|\epsilon_i^p| < \rho$ and $s_i^p + \epsilon_i^p \geq 0$, $\forall\, i \in N$, $p \in \{1, 2\}$.

Given a sequence of best responses to starting vector s, let $r_i^{p,t}(s \mid g)$ denote i's investment in layer p in the t^{th} element of the sequence.

Step 1 : Consider $r^1(s + \boldsymbol{\epsilon} \mid g)$.

In layer g^p, for any $\ell \in L$, if condition 2a holds, then ℓ is connected to at least δ agents in D. After a permutation of s by $\boldsymbol{\epsilon}$, each agent's investment may be reduced by at most ρ. Thus, after such a permutation,

$$
\begin{aligned}
\bar{s}_\ell^p &\geq \delta \hat{s}_\mathrm{d} - k\rho && \text{(C.21)}\\
&> \delta \hat{s}_\mathrm{d} - k\frac{\delta \hat{s}_\mathrm{d} - \hat{s}_\mathrm{s}}{\delta k} && \text{(C.22)}\\
&> \delta \hat{s}_\mathrm{d} - k\frac{\delta \hat{s}_\mathrm{d} - \hat{s}_\mathrm{s}}{k} && \text{(C.23)}\\
&= \hat{s}_\mathrm{s}. && \text{(C.24)}
\end{aligned}
$$

Agent ℓ will therefore make no investment in the first step after a permutation by $\boldsymbol{\epsilon}$.

Alternatively, suppose that 2b holds, and ℓ is connected to at least one agent in I^p and more than one agent in $D \cup I^p$. Then after a permutation of s by $\boldsymbol{\epsilon}$, ℓ's local investment will be such that

$$
\begin{aligned}
\bar{s}_\ell^p &\geq \hat{s}_\mathrm{s} + \hat{s}_\mathrm{d} - k\rho && \text{(C.25)}\\
&> \hat{s}_\mathrm{s} + \hat{s}_\mathrm{d} - k\frac{\hat{s}_\mathrm{d}}{k} && \text{(C.26)}\\
&= \hat{s}_\mathrm{s}. && \text{(C.27)}
\end{aligned}
$$

Again, agent ℓ will make no investment in the first step after a permutation by $\boldsymbol{\epsilon}$. We may thus conclude that $r_\ell^1(s + \boldsymbol{\epsilon} \mid g) = (0, 0)$.

In layer g^1, any $i^2 \in I^2$ is connected to at least one agent in $D \cup I^1$. After a permutation of s by $\boldsymbol{\epsilon}$, it is then the case that $\bar{s}_{i^2}^1 \geq \hat{s}_\mathrm{d} - k\rho$. In layer g^2, i^2 will have previously had no investment of effort by neighbours, and thus permutation by $\boldsymbol{\epsilon}$ results in $\bar{s}_{i^2}^2 \leq k\rho$. This, in turn, implies that i^2's optimal single-action investment in g^2 is weakly greater than $\hat{s}_\mathrm{s} - k\rho$. Then, i^2's marginal payoff from investment in g^1 would be

$$
\begin{aligned}
\frac{\partial \Pi_{i^2}(s + \boldsymbol{\epsilon} \mid g)}{\partial s_{i^2}^1} &\leq f'(\hat{s}_\mathrm{d} - k\rho) - \beta(\hat{s}_\mathrm{s} - k\rho) - c && \text{(C.28)}\\
&< f'(\hat{s}_\mathrm{d} - k\frac{x}{k}) - \beta(\hat{s}_\mathrm{s} - k\frac{x}{k}) - c && \text{(C.29)}\\
&= f'(\hat{s}_\mathrm{d} - x) - \beta(\hat{s}_\mathrm{s} - x) - c && \text{(C.30)}\\
&= 0 && \text{(C.31)}
\end{aligned}
$$

By Lemma C.3, we may then conclude that in the first step after permutation by $\boldsymbol{\epsilon}$, any agent i^2 will be a single-actor investing in layer g^2, and that i^2's investment in g^2 will be at least $\hat{s}_\mathrm{s} - k\rho$. That is, $r_{i^2}^1(s + \boldsymbol{\epsilon} \mid g) = (0, \tilde{s}_{i^2})$, where $\tilde{s}_{i^2} \in [\hat{s}_\mathrm{s} - k, \hat{s}]$ may vary for each member of I^2.

By symmetry, we may conclude as well that $r_{i^1}^1(s + \boldsymbol{\epsilon} \mid g) = (\tilde{s}_{i^1}, 0)$, where $\tilde{s}_{i^1} \in [\hat{s}_\mathrm{s} - k, \hat{s}]$ may vary for each member of I^1.

In layer g^1, any $d \in D$ has no neighbours in $D \cup I^1$. After a permutation of s by $\boldsymbol{\epsilon}$, we know that $\bar{s}_d^p \in [0, k\rho]$. We wish to consider whether it is possible for d to switch to single action. Suppose that d were to become a single-actor in layer g^p. Then,

$$\frac{\partial \Pi_d(s+\boldsymbol{\epsilon} \mid g)}{\partial s_d^1} \geq f'(k\rho) - \beta \hat{s}_{\mathrm{s}} - c \tag{C.32}$$

$$> f'(k\frac{y}{k}) - \beta \hat{s}_{\mathrm{s}} - c \tag{C.33}$$

$$= f'(y) - \beta \hat{s}_{\mathrm{s}} - c \tag{C.34}$$

$$= 0. \tag{C.35}$$

By Lemma C.3, d cannot be a single-actor, and must invest in both layers after a permutation of s by $\boldsymbol{\epsilon}$. If d reduces investment in layer g^p, then marginal payoff increases in layer g^q, which would incentivise d to increase investment in layer g^q. Thus, $\hat{s}_{\mathrm{d}} - k\rho$ represents the minimal amount that d could invest in either layer, and we may write that $r_d^1(s+\boldsymbol{\epsilon} \mid g) = (\tilde{s}_d^1, \tilde{s}_d^2)$, where $\tilde{s}_d^1, \tilde{s}_d^2 \in [\hat{s}_{\mathrm{d}} - k\rho, \hat{s}_{\mathrm{d}}]$ and both values may vary for each member of D.

We will next consider what investment any agent will make after two steps of myopic best responses to the permutation $\boldsymbol{\epsilon}$.

Step 2 : Consider $r^2(\boldsymbol{s}+\boldsymbol{\epsilon}|g)$.

In Step 1, we showed that all free-riders, any agents $\ell \in L$, will make no investment, and single-actors will not invest in the layer in which they were not investing initially. Since dual-actors may only have connections to these two types of agents, we can conclude that after the first step, $\bar{s}_d^p = 0 \,\forall\, p \in \{1, 2\}$. Thus, we conclude that dual-agents will return to their original investments in the second step of myopic best responses, that is $r_d^2(s+\boldsymbol{\epsilon} \mid g) = (\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})$.

Consider any $i^1 \in I^1$. In g^1, i^1 may only be connected to agents in L and I^2. Since none of these agents will invest in layer g^1 in step 1, in step 2 $\bar{s}_{i^1 1} = 0$, and i^1's optimal single-action investment in g^1 is $\hat{s}_{\mathrm{s}}$. Now, suppose that i^1 does make this investment. i^1 is connected to at least one member of $D \cup I^2$ in g^2, and therefore $\bar{s}_{i^1}^2 \leq \hat{s}_{\mathrm{d}} - k\rho$. i^1's marginal payoff in g^2 is

$$\frac{\partial \Pi_{i^1}(s_{Step1} \mid g)}{\partial s_{i^1}^2} \leq f'(\hat{s}_{\mathrm{d}} - k\rho) - \beta \hat{s}_{\mathrm{s}} - c \tag{C.36}$$

$$< f'(\hat{s}_{\mathrm{d}} - k\frac{z}{k}) - \beta \hat{s}_{\mathrm{s}} - c \tag{C.37}$$

$$= f'(y) - \beta \hat{s}_{\mathrm{s}} - c \tag{C.38}$$

$$= 0. \tag{C.39}$$

Thus, $r_{i^1}^2(s+\boldsymbol{\epsilon} \mid g) = (\hat{s}_{\mathrm{s}}, 0)$ satisfies agent i's first order conditions, and by Lemma C.1, we may conclude that any agent $i^1 \in I^1$ will make this investment.

By symmetry, we conclude as well that $r^2_{i^2}(s+\boldsymbol{\epsilon} \mid g) = (0, \hat{s}_{\mathrm{s}})$.

Finally, consider any agent $\ell \in L$. If 2b holds, then ℓ will be connected to at least one investing single-actor and at least one dual-actor in either layer g^p. Then

$$\begin{aligned}
\bar{s}^p_\ell &\geq (\hat{s}_{\mathrm{s}} - k\rho) + (\hat{s}_{\mathrm{d}} - k\rho) && \text{(C.40)}\\
&> \hat{s}_{\mathrm{s}} - \hat{s}_{\mathrm{d}} - 2k\frac{\hat{s}_{\mathrm{d}}}{k} && \text{(C.41)}\\
&= \hat{s}_{\mathrm{s}} + \hat{s}_{\mathrm{d}} && \text{(C.42)}\\
&> \hat{s}_{\mathrm{s}}. && \text{(C.43)}
\end{aligned}$$

So ℓ will not invest in either layer.

Now, consider the case when 2a holds, then ℓ is connected to at least δ dual-actors in either layer g^p. In this case,

$$\begin{aligned}
\bar{s}^p_\ell &\geq \delta(\hat{s}_{\mathrm{d}} - k\rho) && \text{(C.44)}\\
&> \delta\hat{s}_{\mathrm{d}} - \delta k\frac{\delta\hat{s}_{\mathrm{d}} - \hat{s}_{\mathrm{s}}}{\delta k} && \text{(C.45)}\\
&= \hat{s}_{\mathrm{s}}. && \text{(C.46)}
\end{aligned}$$

Thus, ℓ will not invest in either layer. We may then conclude that $r^2_\ell(s+\boldsymbol{\epsilon} \mid g) = (0,0)$.

Thus, we have shown that all agents will return to their original investments after a permutation of s by $\boldsymbol{\epsilon}$ and two steps of myopic best responses.

Now, we will prove the opposite direction, that an equilibrium that is stable must be characterised as set out in Theorem 3.2. This requires the following lemma.

Lemma C.5. *Let $s, s' \in S$ be two distinct action profiles such that $\forall\, i \in N$ $s^1_i \geq s^{1\prime}_i$, and $s^2_i \leq s^{2\prime}_i$. Then, $\forall\, i \in N$, $r^{1,2}_i(s \mid g) \geq r^{1,2}_i(s' \mid g)$ and $r^{2,2}_i(s \mid g) \leq r^{2,2}_i(s' \mid g)$.*

As this lemma is notationally dense, we will state it in words. Given the two action profiles $s, s' \in S$, if the actions of all agents in layer g^1 are weakly greater in s than in s', and the actions of all agents in layer g^2 are weakly greater in s' than in s, then after two steps of simultaneous myopic best responses by all agents the same relationships will hold.

Proof of Lemma C.5. Let $s, s' \in S$ be two separate action profiles such that $\forall\, i \in N$, $s^1_i \geq s^{1\prime}_i$ and $s^2_i \leq s^{2\prime}_i$.

Pick any arbitrary agent $i \in N$, and suppose that $r^1_i(s) = (\hat{s}_{\mathrm{s}} - \bar{s}^1_i, 0)$. Then we know that $f'(\bar{s}^2_i) - \beta(\hat{s}_{\mathrm{s}} - \bar{s}^1_i) - c \leq 0$ by Lemma C.3. But then

$$\begin{aligned}
f'\left(\bar{s}^{2\prime}_i\right) - \beta\left(\hat{s}'_{\mathrm{s}} - \bar{s}^{1\prime}_i\right) - c &\leq f'\left(\bar{s}^2_i\right) - \beta\left(\hat{s}_{\mathrm{s}} - \bar{s}^1_i\right) - c && \text{(C.47)}\\
&\leq 0, && \text{(C.48)}
\end{aligned}$$

so $r_i^1(s') = \left(\hat{s}'_{\mathrm{s}} - \bar{s}_i^{1\prime}, 0\right)$, and because $\bar{s}_i^{1\prime} \leq \bar{s}_i^{1\prime}$, i's investment in g^1 in $r_i^1(s')$ must be weakly larger than i's investment in $r_i^1(s)$, and thus $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$.

By a similar argument, it can be shown that if $r_i^1(s') = \left(0, \hat{s}_{\mathrm{s}} - \bar{s}_i^{2\prime}\right)$, then it must be the case that $r_i^1(s) = (0, \hat{s}_{\mathrm{s}} - \bar{s}_i^2)$, and so $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$.

Because a single-actor will always have a greater marginal payoff function in the layer of investment than a dual-actor, if $r_i^1(s') = \left(\hat{s}'_{\mathrm{s}} - \bar{s}_i^{1\prime}, 0\right)$ and $r_i^1(s)$ involves dual-investment, or if $r_i^1(s) = (0, \hat{s}_{\mathrm{s}} - \bar{s}_i^2)$ and $r_i^1(s')$ involves dual-investment, then the relationships $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$ must hold.

Suppose that $r_i(s) = (0, \hat{s}_{\mathrm{s}} - \bar{s}_i^2)$ and $r_i(s') = (\hat{s}_{\mathrm{s}} - \bar{s}_i^{1\prime}, 0)$. Then clearly $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$.

We have now shown for all boundary solutions that $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$, which leaves only the case when both $r_i^1(s)$ and $r_i^1(s')$ involve dual-investment.

We know that the system

$$f'(\bar{x} + x) - \beta y - c = 0 \text{ and} \tag{C.49}$$

$$f'(\bar{y} + y) - \beta x - c = 0 \tag{C.50}$$

is unique, due to Lemma C.1. Now we consider what happens to the solution when $\bar{x}$ changes and $\bar{y}$ remains constant. From Equation (C.50),

$$f''(\bar{y} + y)\,\mathrm{d}y - \beta\,\mathrm{d}x = 0 \tag{C.51}$$

$$\mathrm{d}y = \frac{\beta}{f''(\bar{y} + y)}\,\mathrm{d}x. \tag{C.52}$$

Then plug this into the total derivative of Equation (C.49)

$$f''(\bar{x} + x)(\mathrm{d}\bar{x} + \mathrm{d}x) - \beta\frac{\beta}{f''(\bar{y} + y)}\,\mathrm{d}x = 0 \tag{C.53}$$

$$f''(\bar{x} + x)\left(1 + \frac{\mathrm{d}x}{\mathrm{d}\bar{x}}\right) - \frac{\beta^2}{f''(\bar{y} + y)}\frac{\mathrm{d}x}{\mathrm{d}\bar{x}} = 0 \tag{C.54}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\bar{x}} = \frac{-f''(\bar{x} + x)f''(\bar{y} + y)}{f''(\bar{x} + x)f''(\bar{y} + y) - \beta^2}. \tag{C.55}$$

Because of Assumption 3.1, we know that the denominator of Equation (C.55) is positive, while the numerator is negative. Thus, we may conclude that, at the unique solution, $\frac{\mathrm{d}x}{\mathrm{d}\bar{x}} < 0$, which in turn implies that $\frac{\mathrm{d}y}{\mathrm{d}\bar{x}} > 0$, from Equation (C.52). If $\bar{x}$ increases and $\bar{y}$ decreases, sequential application of this conclusion ensures that the optimal value of x will decrease and the optimal value of y will increase.

We know that $\bar{s}_i^1 \geq \bar{s}_i^{1\prime}$ and $\bar{s}_i^2 \leq \bar{s}_i^{2\prime}\,\forall\, i \in N$. If both $r_i^1(s)$ and $r_i^1(s')$ involve dual-investment, then from Equations (C.52) and (C.55) we may conclude that $r_i^{1,1}(s) \leq r_i^{1,1}(s')$ and $r_i^{2,1}(s) \geq r_i^{2,1}(s')$, and we can therefore conclude that these two relationships always hold.

But then, by applying this fact twice, it must be true that a second step of best-response function r yields $r_i^{1,2}(s) \geq r_i^{1,2}(s')$ and $r_i^{2,2}(s) \leq r_i^{2,2}(s')$ $\forall\, i \in N$, which proves this lemma.

□

Now, suppose that $s^* \in S$ is a stable equilibrium, and suppose there exists an investor i such that $\bar{s}_i^1 \leq \bar{s}_i^2 < \hat{s}_{\mathrm{s}}$, and $\hat{s}_{\mathrm{s}} - \bar{s}_i^1 < \frac{f'(\bar{s}_i^2)-c}{\beta}$. Proposition 3.1 tells us that i will be making an interior investment in both layers. Let i's investment be $s_i = (s_i^1, s_i^2)$. Because i's investment is interior, there must be some $\delta > 0$ such that $\tilde{s}_i = (s_i^1 + \delta, s_i^2) \in \tilde{S}$. Now, suppose s^* is permuted such that i now invests $\tilde{s}_i$ and no other agent changes their investment. By Lemma C.5, in every second iteration of the best response function, i's investment in g^1 is greater than $s_i^1 + \delta$. Therefore, a sequence of best responses to this permutation will not converge to s^*. Thus, there can be no agent i making an interior investment in both layers.

Now, suppose there exists an investor i such that $\bar{s}_i^1 \leq \bar{s}_i^2 < \hat{s}_{\mathrm{s}}$, and $\hat{s}_{\mathrm{s}} - \bar{s}_i^1 \geq \frac{f'(\bar{s}_i^2)-c}{\beta}$. Proposition 3.1 tells us that i will be making an investment only in layer g^1. Let i's investment be $s_i = (s_i^1, 0)$, with $s_i^1 < \hat{s}_{\mathrm{s}}$. Then, there must be some $\delta > 0$ such that $s_i^1 + \delta < \hat{s}_{\mathrm{s}}$. Now, suppose s^* is permuted such that i now invests makes investment $(s_i^1 + \delta, 0)$ and no other agent changes their investment. By Lemma C.5, in every second iteration of the best response function, i's investment in g^1 is greater than $s_i^1 + \delta$. Therefore, a sequence of best responses to this permutation will not converge to s^*. Thus, there can be no agent i making investment s_i.

We have shown that there may be no interior investors in a stable equilibrium, leaving only specialists investing $(0,0)$, $(\hat{s}_{\mathrm{s}}, 0)$, $(0, \hat{s}_{\mathrm{s}})$, or $(\hat{s}_{\mathrm{d}}, \hat{s}_{\mathrm{d}})$. According to the criteria established in Proposition 3.1, such investments will only be best responses when the set of all agents may be partitioned into the sets L, I^1, I^2, and D.

□

Proof of Proposition 3.3. I have claimed that the function

$$\phi(s) = \begin{pmatrix} \alpha^1 \mathbf{1} \\ \alpha^2 \mathbf{1} \end{pmatrix}^{\mathsf{T}} s - \frac{1}{2} s^{\mathsf{T}} \left(\boldsymbol{I} - \begin{bmatrix} \delta^1 \boldsymbol{G}^1 & \beta \boldsymbol{I} \\ \beta \boldsymbol{I} & \delta^2 \boldsymbol{G}^2 \end{bmatrix} \right) s \tag{C.56}$$

satisfies the properties of a potential function. That is, $\forall\, s_i, s_i'$ and $\forall\, i \in N$,

$$\phi(s_i, s_{-i}) - \phi\left(s_i', s_{-i}\right) = \Pi_i(s_i, s_{-i} \mid g) - \Pi_i\left(s_i', s_{-i} \mid g\right). \tag{C.57}$$

Then, $\forall\, s, s' \in S$ such that for any $i \in N$, $s_i \neq s_i'$ implies $s_j = s_j' \,\forall\, j \neq i$,

$$
\begin{aligned}
&\phi(s_i, s_{-i}) - \phi\left(s_i', s_{-i}\right) && \text{(C.58)}\\
=&\phi(s) - \phi\left(s'\right) && \text{(C.59)}\\
=&\begin{pmatrix}\alpha^1\mathbf{1}\\ \alpha^2\mathbf{1}\end{pmatrix}^{\mathsf{T}} s - \frac{1}{2}s^{\mathsf{T}}\left(\boldsymbol{I} - \begin{bmatrix}\delta^1\boldsymbol{G}^1 & \beta\boldsymbol{I}\\ \beta\boldsymbol{I} & \delta^2\boldsymbol{G}^2\end{bmatrix}\right)s \\
&\quad - \begin{pmatrix}\alpha^1\mathbf{1}\\ \alpha^2\mathbf{1}\end{pmatrix}^{\mathsf{T}} s' - \frac{1}{2}s'^{\mathsf{T}}\left(\boldsymbol{I} - \begin{bmatrix}\delta^1\boldsymbol{G}^1 & \beta\boldsymbol{I}\\ \beta\boldsymbol{I} & \delta^2\boldsymbol{G}^2\end{bmatrix}\right)s' && \text{(C.60)}\\
=&\sum_{j\in N}\left[\alpha^1 s_j^1 + \alpha^2 s_j^2 - \frac{1}{2}\left(\left(s_j^1\right)^2 + \left(s_j^2\right)^2\right) + \beta s_j^1 s_j^2\right] \\
&\quad + \frac{1}{2}\left[\sum_{j\in N}\sum_{k\in N}(\delta^1 g_{jk}^1 s_j^1 s_k^1 + \delta^2 g_{jk}^2 s_j^2 s_k^2)\right] \\
&\quad - \left\{\sum_{j\in N}\left[\alpha^1 s_j^{1\prime} + \alpha^2 s_j^{2\prime} - \frac{1}{2}\left(\left(s_j^{1\prime}\right)^2 + \left(s_j^{2\prime}\right)^2\right) + \beta s_j^{1\prime} s_j^{2\prime}\right]\right. \\
&\quad \left. + \frac{1}{2}\left[\sum_{j\in N}\sum_{k\in N}(\delta^1 g_{jk}^1 s_j^{1\prime} s_k^{1\prime} + \delta^2 g_{jk}^2 s_j^{2\prime} s_k^{2\prime})\right]\right\} && \text{(C.61)}\\
=&\,\alpha^1 s_i^1 + \alpha^2 s_i^2 - \frac{1}{2}\left(\left(s_i^1\right)^2 + \left(s_i^2\right)^2\right) + \beta s_i^1 s_i^2 + \delta^1 s_i^1 \sum_{j\in N} g_{ij}^1 s_j^1 + \delta^2 s_i^2 \sum_{j\in N} g_{ij}^2 s_j^2 \\
&\quad - \left[\alpha^1 s_i^{1\prime} + \alpha^2 s_i^{2\prime} - \frac{1}{2}\left(\left(s_i^{1\prime}\right)^2 + \left(s_i^{2\prime}\right)^2\right) + \beta s_i^{1\prime} s_i^{2\prime}\right. \\
&\quad \left. + \delta^1 s_i^{1\prime} \sum_{j\in N} g_{ij}^1 s_j^{1\prime} + \delta^2 s_i^{2\prime} \sum_{j\in N} g_{ij}^2 s_j^{2\prime}\right] && \text{(C.62)}\\
=&\Pi_i(s_i, s_{-i} \mid g) - \Pi_i\left(s_i', s_{-i} \mid g\right) && \text{(C.63)}
\end{aligned}
$$

Thus, ϕ satisfies the properties of a potential function.

□

Proof of Theorem 3.3. Let

$$
\upsilon^p = \begin{cases}\delta^p\lambda_{\max}(\boldsymbol{G}^p) & \text{if } \delta^p > 0\\ \delta^p\lambda_{\min}(\boldsymbol{G}^p) & \text{if } \delta^p < 0.\end{cases} \tag{C.64}
$$

If

$$
\beta^2 < (1-\upsilon^1)(1-\upsilon^2), \tag{C.65}
$$

then the game $\Gamma(N, S, g)$ has a unique equilibrium on the action space S.

From Proposition 3.3, we know that the potential function for the game $\Gamma(N, S, g)$ is

$$\phi(s) = \begin{pmatrix} \alpha^1 \mathbf{1} \\ \alpha^2 \mathbf{1} \end{pmatrix}^{\mathsf{T}} s - \frac{1}{2} s^{\mathsf{T}} \left(\boldsymbol{I} - \begin{bmatrix} \delta^1 \boldsymbol{G}^1 & \beta \boldsymbol{I} \\ \beta \boldsymbol{I} & \delta^2 \boldsymbol{G}^2 \end{bmatrix} \right) s. \tag{C.66}$$

Then,

$$\frac{\partial \phi(s)}{\partial s} = \begin{pmatrix} \alpha^1 \mathbf{1} \\ \alpha^2 \mathbf{1} \end{pmatrix} - \left(\boldsymbol{I} - \begin{bmatrix} \delta^1 \boldsymbol{G}^1 & \beta \boldsymbol{I} \\ \beta \boldsymbol{I} & \delta^2 \boldsymbol{G}^2 \end{bmatrix} \right) s \tag{C.67}$$

and

$$\nabla \phi(s) = - \left(\boldsymbol{I} - \begin{bmatrix} \delta^1 \boldsymbol{G}^1 & \beta \boldsymbol{I} \\ \beta \boldsymbol{I} & \delta^2 \boldsymbol{G}^2 \end{bmatrix} \right). \tag{C.68}$$

To prove that $\phi(s)$ has a unique global maximum on S, it is sufficient to show that $-\nabla\phi(s)$ is positive definite on S. First, let

$$\boldsymbol{M} = -\nabla \phi(s) = \begin{bmatrix} \boldsymbol{I} - \delta^1 \boldsymbol{G}^1 & -\beta \boldsymbol{I} \\ -\beta \boldsymbol{I} & \boldsymbol{I} - \delta^2 \boldsymbol{G}^2 \end{bmatrix}. \tag{C.69}$$

$\boldsymbol{M}$ is positive definite if and only if its upper left block and the Schur complement if its upper left block are positive definite. These are $(\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)$ and $(\boldsymbol{I} - \delta^2 \boldsymbol{G}^2) - \beta^2 (\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)^{-1}$ respectively.

Recall that

$$\upsilon^p = \begin{cases} \delta^p \lambda_{\max}(\boldsymbol{G}^p) & \text{if } \delta^p > 0 \\ \delta^p \lambda_{\min}(\boldsymbol{G}^p) & \text{if } \delta^p < 0. \end{cases} \tag{C.70}$$

Thus, $-\upsilon^1$ is the minimum eigenvalue of $-\delta^1 \boldsymbol{G}^1$, which requires the fact that the minimum eigenvalue of an adjacency matrix is less than or equal to 0. It follows, then, that $1 - \upsilon^1$ is the minimum eigenvalue of $(\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)$. We may thus conclude that $(\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)$ is positive definite if and only if $1 - \upsilon^1 > 0$.

Consider now the Schur complement, $(\boldsymbol{I} - \delta^2 \boldsymbol{G}^2) - \beta^2 (\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)^{-1}$. By Weyl's inequality,

$$\lambda_{\min} \left[(\boldsymbol{I} - \delta^2 \boldsymbol{G}^2) - \beta^2 (\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)^{-1} \right] \tag{C.71}$$
$$\geq \lambda_{\min}(\boldsymbol{I} - \delta^2 \boldsymbol{G}^2) + \lambda_{\min}(-\beta^2 (\boldsymbol{I} - \delta^1 \boldsymbol{G}^1)^{-1}) \tag{C.72}$$
$$= (1 - \upsilon^2) - \frac{\beta^2}{1 - \upsilon^1}. \tag{C.73}$$

Now, we have assumed

$$\beta^2 < (1 - \upsilon^1)(1 - \upsilon^2) \tag{C.74}$$
$$\iff (1 - \upsilon^2) - \frac{\beta^2}{1 - \upsilon^1} > 0, \tag{C.75}$$

which is sufficient to show that the minimum eigenvalue of the Schur complement is greater than zero, and thus the Schur complement is positive definite.

Thus, the assumptions that $1 - \upsilon^1 > 0$ and $\beta^2 < (1 - \upsilon^1)(1 - \upsilon^2)$ are sufficient to ensure that there is a unique equilibrium.

$\square$

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Nomenclature

LHS	Left-hand side
RHS	Right-hand side
SPE	Subgame-perfect-Nash equilibrium

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